

# An Analytical Framework for Assessing Asset Pricing Models and Predictability \*

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## Abstract

New insights about the connections between stock market volatility and returns, the pricing of long-run claims, or return predictability have recently revived interest in consumption-based equilibrium asset pricing. The recursive utility model is prominently used in these contexts to determine the price of assets in equilibrium. Often, solutions are approximate and quantities of interest are computed through simulations. We propose an approach that delivers closed-form formulas for price-consumption and price-dividend ratios, as well as for many of the statistics usually computed to assess the ability of the model to reproduce stylized facts. The proposed framework is flexible enough to capture rich dynamics for consumption and dividends. Closed-form formulas facilitate the economic interpretation of empirical results. We illustrate the usefulness of our approach by investigating the properties of long-run asset pricing models in many empirical dimensions.

## 1 Introduction

In the last twenty years or so, financial economists have devoted a lot of energy to solving two unyielding puzzles, the equity premium puzzle and the risk-free rate puzzle. The specification of preferences in the basic consumption CAPM model introduced by Lucas (1978) and Breeden (1979) has been modified to accommodate a large equity premium and a rather low risk-free rate. The recursive utility model of Epstein and Zin (1989, 1991) is one such extension that allows to disentangle attitudes towards risk and preferences for intertemporal substitution of consumption. Recently, this model has been geared towards reproducing new facts about the connections between stock market volatility and returns, the pricing of long-run claims, or return predictability (see in particular Bansal and Yaron, 2004, Bansal, Gallant and Tauchen, 2004, Hansen, Heaton and Li, 2004, Lettau, Ludvigson and Wachter, 2004).

The key ingredient in explaining these new facts is the specification of the endowment process. New joint dynamic models have been proposed for consumption and dividend growth, while in early models the equality of consumption and dividend was often assumed. Often, these new full-fledged models necessitate a numerical resolution or log-linear approximations to find the equilibrium price-consumption and price-dividend ratios. Moreover, to assess the model, several statistics such as the first and second moments of asset returns or coefficients in predictability regressions are computed through simulations. Whenever preference parameters are changed, a new numerical solution has to be found and simulations run again to compute the statistics of interest. Given this time-consuming process, only few parameter configurations are explored.

Our first and main contribution is to propose an approach that solves analytically this model. We begin by providing formulas for the price-consumption, the price-dividend and the risk-free bond in the recursive utility models of Epstein and Zin (1989) and Hansen, Heaton and Li (2004). A key assumption to obtain these analytical solutions is that the logarithms of real per capita consumption and dividend growth follow a bivariate process where both the means, variances and covariances change according to a Markov variable  $s_t$  which takes the values 1 to  $N$ , the postulated number of states in the economy, where  $s_t$  is a stationary and homogenous Markov chain.

Markov switching models have been used previously in the asset pricing literature to capture the dynamics of the endowment process but few papers have derived analytical pricing expressions. This paper extends considerably the closed-form pricing formulas provided in Bonomo and Garcia (1994) and Cecchetti, Lam and Mark (1990) for the Lucas (1978) CCAPM model. For recursive preferences, solutions to the Euler equations have

been found either numerically or after a log linear transformation.

A second important contribution is the possibility to compute analytically many of the statistics that researchers have attempted to reproduce with the postulated asset pricing model: the first and second moments of the equity premium and of the risk-free rate, the mean of and the volatility of the price-dividend ratio, the predictability of returns and excess returns by the dividend-price ratio, the predictability of consumption volatility by the dividend-price ratio and the consumption-wealth ratio, as well as the negative autocorrelation of returns and excess returns at long horizons. We provide formulas for all these statistics. It should be noted that the statistics will differ between models only insofar as the asset prices given by the models are different; otherwise the expressions are identical.

A third useful contribution is to use the exact pricing formulas to assess the impact of approximations that researchers usually apply to solve asset pricing models. One pervasive approximation is the log-linearization of Campbell and Shiller (1988). More recently, Hansen, Heaton and Li (2004) proposed a Taylor expansion-based approximation around a set value of the elasticity of substitution parameter in a recursive utility model. Using our exact pricing formulas, we are able to say in which regions of the parameter space some approximations behave better than others.

Finally, to illustrate the usefulness of our analytical framework, we apply it to two prominent recent models by Lettau, Ludvigson and Wachter (2004) and Bansal and Yaron (2004). Both promote the role of macroeconomic uncertainty measured by the volatility of consumption as a determining factor in the pricing of assets. The first paper models consumption growth as a Markov switching process and uses Epstein and Zin (1989) preferences, and so fits directly in our framework. The second paper uses the same preferences but models the consumption-dividend endowment as an autoregressive process with time-varying volatility. We set the latter model in our Markov-switching framework and are able to compare the two models in terms of asset pricing and predictability implications. Our analytical formulas allow us to explore a much wider set of preference parameters than in the original papers and thus to better understand their role in determining the financial quantities of interest.

The importance of deriving closed-form formulas should not be underestimated. Lettau, Ludvigson and Wachter (2004), who use precisely a Markov-switching model for their endowment, remark that their two-state model takes very long to solve and that a three-state model would be computationally infeasible. They use a learning model that they must solve at each time period given their new assessment of the transition probabilities of the Markov process. Our formulas can be adapted to this approach and will ease considerably the process. Another considerable saving of processing time comes potentially from the

simulations researchers run to compute predictability regressions. The usual procedure is to try to replicate the actual statistics with the same number of observations as in the sample as well with a much larger number of observations to see if the model can produce predictability in population. The last exercise, the most costly in computing time, is avoided by using the formulas we provide. The same is true for the variance ratios.

Recently, some papers have also proposed to develop analytical formulas for asset pricing models. Abel (2005) calculates exact expressions for risk premia, term premia, and the premium on levered equity in a framework that includes habit formation and consumption externalities (keeping up or catching up with the Joneses). The formulas are derived under lognormality and an i.i.d. assumption for the growth rates of consumption and dividends. We also assume lognormality but after conditioning on a number of states and our state variable capture the dynamics of the growth rates. Eraker (2006) produces analytic pricing formulas for stocks and bonds in an equilibrium consumption CAPM with Epstein-Zin preferences, under the assumption that consumption and dividend growth rates follow affine processes. However, he uses the Campbell and Shiller (1988) approximation to maintain a tractable analytical form of the pricing kernel. Quite recently, Gabaix (2007) proposed a class of linearity-generating processes that ensures closed-form solutions for the prices of stocks and bonds. This solution strategy is based on reverse-engineering of the processes for the stochastic discount factors and the asset payoffs.

The rest of the paper is organized as follows. Section 2 describes the Markov-switching model for consumption and dividend growth. In Section 3, we solve for the price-consumption, the price-dividend ratios and the risk-free bond in asset pricing models. Section 4 enumerates several empirical facts used to gauge the validity of asset pricing models and provides analytical formulas for the statistics reproducing these stylized facts. Section 5 provides applications to several asset pricing models for the US post-war economy. Section 6 concludes. A technical appendix collects the set of formulas used in the empirical part.

## **2 A Markov-Switching Model for Consumption and Dividends**

We follow the approach pioneered by Mehra and Prescott (1985) by specifying a stochastic process for the endowment process and solving the model for the prices of the market portfolio, an equity and the risk-free asset in the economy. The goal in this branch of the empirical asset pricing literature is to determine if equilibrium models with reasonable preferences are able to reproduce some stylized facts associated with returns, consumption and dividends.

Contrary to the original model in Lucas (1978)), we make a distinction between consumption and dividends. Consumption is the payoff on the market portfolio while dividends

accrue to equity owners. This distinction is nowadays almost always made (see Bansal and Yaron, 2004, Hansen, Heaton and Li (2004) and Lettau, Ludvigson and Wachter (2005) among others), but was introduced originally by Tauchen (1986) and pursued further by Cecchetti, Lam and Mark (1993) and Bonomo and Garcia (1994, 1996).<sup>1</sup>

The main reason for disentangling the consumption and dividend processes is first and foremost an empirical one: the series are very different in terms of mean, variance, and other moments. We postulate that the logarithms of consumption and dividends growth follow a bivariate process where both the means, variances and covariances change according to a vector Markov variable  $\zeta_t$  which takes  $N$  values (when  $N$  states of nature are assumed for the economy):

$$\zeta_t = \begin{cases} (1, 0, 0, \dots, 0)^\top & \text{for state 1} \\ (0, 1, 0, \dots, 0)^\top & \text{for state 2} \\ \vdots & \vdots \\ \vdots & \vdots \\ (0, 0, 0, \dots, 1)^\top & \text{for state } N. \end{cases}$$

The sequence of vector Markov variables evolves according to a transition probability matrix  $P$  defined as:

$$P^\top = [p_{ij}]_{1 \leq i, j \leq N}, \quad p_{ij} = P(\zeta_{t+1} = e_j \mid \zeta_t = e_i). \quad (2.1)$$

where  $e_i$  is a column vector with zeroes everywhere except in the  $i^{\text{th}}$  position, which has the value one. We assume that the Markov chain is stationary with an ergodic distribution  $E[\zeta_t] = \Pi \in \mathbb{R}^N$ . For the second moments we have:

$$E[\zeta_t \zeta_t^\top] = \text{Diag}(\Pi_1, \dots, \Pi_N) \quad \text{and} \quad \text{Var}[\zeta_t] = \text{Diag}(\Pi_1, \dots, \Pi_N) - \Pi \Pi^\top. \quad (2.2)$$

Therefore, the logarithms of consumption and dividends growth can be written as follows:

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<sup>1</sup>Abel (1992) formulates a model with production, but where the labor supply is inelastic and the stock of capital is fixed and does not depreciate, and randomness comes from technology shocks. Then, consumption is equal to the total income of the economy, which is the sum of dividends - the capital income - with labor income. The disentanglement of consumption and dividends appears naturally in an asset pricing model of a production economy. However, total income is usually different from consumption, since there is investment, and although the Euler condition for asset returns still involves discounting the return by the intertemporal marginal rate of substitution in consumption, the latter depends also on leisure (see Brock, 1982, and Danthine and Donaldson, 1995). In Abel's (1992) simple version, labour supply is fixed and there is no investment. Thus, his version of a production economy fits perfectly our empirical framework.

$$x_{c,t+1} \equiv \log(C_{t+1}) - \log(C_t) = c_{t+1} - c_t = \mu_c^\top \zeta_t + (\omega_c^\top \zeta_t)^{1/2} \varepsilon_{c,t+1} \quad (2.3)$$

$$x_{d,t+1} \equiv \log(D_{t+1}) - \log(D_t) = d_{t+1} - d_t = \mu_d^\top \zeta_t + (\omega_d^\top \zeta_t)^{1/2} \varepsilon_{d,t+1}, \quad (2.4)$$

where

$$\begin{aligned} & \begin{pmatrix} \varepsilon_{c,t+1} \\ \varepsilon_{d,t+1} \end{pmatrix} \mid \sigma(\varepsilon_{c,\tau}, \varepsilon_{d,\tau}, \tau \leq t; \zeta_m, m \in \mathbf{Z}) \\ & \sim \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \rho^\top \zeta_t \\ \rho^\top \zeta_t & 1 \end{bmatrix} \right). \end{aligned} \quad (2.5)$$

Bonomo and Garcia (1994, 1996) use the specification (2.3,2.4) with constant correlations for the joint consumption-dividends process to investigate if an equilibrium asset pricing model with different types of preferences can reproduce various features of the real and excess return series.<sup>2</sup> The heteroscedasticity of the endowment process measures economic uncertainty as put forward by Bansal and Yaron (2004).

### 3 Solving The Recursive Utility Model of Asset Pricing

In this widely used asset pricing model introduced by Epstein and Zin (1989), a representative agent derives his utility by combining current consumption with a certainty equivalent of future utility through an aggregator. Depending on how this certainty equivalent is specified, the recursive utility concept can accommodate several classes of preferences. A class that is used extensively used in empirical work is the so-called Kreps-Porteus, where the certainty equivalent conforms with expected utility for ranking timeless gambles, but with a different parameter than the aggregator's parameter. This is what it is usually called the Epstein and Zin (1989) model. We will keep below with this tradition.<sup>3</sup>

The main goal of this section is to characterize the price-consumption ratio  $P_{M,t}/C_t$  (the price of the unobservable market portfolio that pays off consumption), the price-dividend ratio  $P_t/D_t$  (the price of an asset that pays the aggregate dividend), and finally the price  $P_{f,t}/1$  of a risk-free bond that pays for sure one unit of consumption.

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<sup>2</sup>Cecchetti, Lam, and Mark (1991) use a two-state homoskedastic specification of (11) for the endowment and similar preferences to try to match the first and second moments of the return series. The authors use two models, one with a leverage economy, another with a pure exchange economy without bonds. In both instances, they are unable to replicate the first and second moments taken together.

<sup>3</sup>Epstein and Zin (1989) go further by integrating in a temporal setting a large class of atemporal non-expected utility theories, in particular homogeneous members of the class introduced by Chew (1985) and Dekel (1986). The certainty equivalent is then defined implicitly. It includes in particular a disappointment aversion specification, see Bonomo and Garcia (1994).

### 3.1 The Recursive Utility Model

The representative agent has recursive utility defined over consumption flow  $C_t$  as follows:

$$V_t = \left\{ (1 - \delta) C_t^{1 - \frac{1}{\psi}} + \delta [\mathcal{R}_t(V_{t+1})]^{1 - \frac{1}{\psi}} \right\}^{\frac{1}{1 - \frac{1}{\psi}}} \quad \text{if } \psi \neq 1 \quad (3.1)$$

$$= C_t^{1 - \delta} [\mathcal{R}_t(V_{t+1})]^\delta \quad \text{if } \psi = 1, \quad (3.2)$$

where  $V_t$  is the current continuation value of investor utility,  $\mathcal{R}_t(V_{t+1}) = (E[V_{t+1}^{1-\gamma} | J_t])^{\frac{1}{1-\gamma}}$  is the certainty equivalent of the next period continuation value of investor utility,  $\gamma$  is the coefficient of relative risk aversion,  $\psi$  is the elasticity of intertemporal substitution,  $\delta$  is the subjective discount factor and  $\theta = (1 - \gamma) / (1 - 1/\psi)$ .

When  $\psi \neq 1$ , the stochastic discount factor is given by:

$$M_{t,t+1} = \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \delta \frac{\frac{P_{M,t+1}}{C_{t+1}} + 1}{\frac{P_{M,t}}{C_t}} \right)^{\theta-1}, \quad (3.3)$$

where the market price-consumption ratio is given by:

$$\frac{P_{M,t}}{C_t} = \delta E \left[ \left( \frac{P_{M,t+1}}{C_{t+1}} + 1 \right)^\theta \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \mid J_t \right]^{\frac{1}{\theta}}. \quad (3.4)$$

If  $\theta = 1$ , one remarks that (3.3) corresponds to the stochastic discount factor of an investor with time-separable utility and constant relative risk aversion.

When  $\psi = 1$ , the benchmark case considered by Hansen, Heaton and Li (2004), the stochastic discount factor is given by:

$$M_{t,t+1}^1 = \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma} \left( \frac{\frac{V_{t+1}^1}{C_{t+1}}}{\left( \frac{V_t^1}{C_t} \right)^{\frac{1}{\delta}}} \right)^{1-\gamma}. \quad (3.5)$$

where the utility-consumption ratio is given by:

$$\frac{V_t^1}{C_t} = \left( E \left[ \left( \frac{V_{t+1}^1}{C_{t+1}} \right)^{1-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \mid J_t \right] \right)^{\frac{\delta}{1-\gamma}} \quad (3.6)$$

whereas the market price-consumption ratio is constant and equal to  $\delta / (1 - \delta)$ .

One important property that we will use in deriving our analytical formulas is the Markov property of the model. We will show that the variables  $P_t/D_t$ ,  $P_{M,t}/C_t$ ,  $P_{f,t}/1$  and  $V_t^1/C_t$  are (non-linear) functions of the state variable  $\zeta_t$ . On the other hand, the state  $\zeta_t$  takes a finite number of values. Consequently, any real non-linear function  $g(\cdot)$  of  $\zeta_t$  is a



linear function of  $\zeta_t$ .<sup>4</sup> This property will allow us to characterize analytically the price-payoff ratios while other data generating processes need either linear approximations or numerical methods to solve the model. The structure of the endowment process implies that there will be one such payoff-price ratio per regime and this will help in computing closed-form analytical formulas.

For these price-payoff ratios, we adopt the following notation:

$$\frac{P_t}{D_t} = \lambda_1^\top \zeta_t, \quad (3.7)$$

$$\frac{P_{M,t}}{C_t} = \lambda_{1c}^\top \zeta_t, \quad (3.8)$$

$$\frac{P_{f,t}}{1} = \lambda_{1f}^\top \zeta_t. \quad (3.9)$$

$$\frac{V_t^1}{C_t} = z^\top \zeta_t. \quad (3.10)$$

Observe also that one can write

$$\frac{D_t}{P_t} = \lambda_2^\top \zeta_t \quad \text{with} \quad \lambda_2 = (\lambda_{11}^{-1}, \dots, \lambda_{1N}^{-1})^\top, \quad \text{where} \quad \lambda_1 = (\lambda_{11}, \dots, \lambda_{1N})^\top. \quad (3.11)$$

Likewise,

$$\frac{C_t}{P_{M,t}} = \lambda_{2c}^\top \zeta_t \quad \text{with} \quad \lambda_{2c} = (\lambda_{1c1}^{-1}, \dots, \lambda_{1cN}^{-1})^\top, \quad \text{where} \quad \lambda_{1c} = (\lambda_{1c1}, \dots, \lambda_{1cN})^\top, \quad (3.12)$$

$$R_{f,t+1} = \frac{1}{P_{f,t}} = \lambda_{2f}^\top \zeta_t \quad \text{with} \quad \lambda_{2f} = (\lambda_{1f1}^{-1}, \dots, \lambda_{1fN}^{-1})^\top, \quad \text{where} \quad \lambda_{1f} = (\lambda_{1f1}, \dots, \lambda_{1fN})^\top. \quad (3.13)$$

Solving the Epstein-Zin model amounts to characterize the vectors  $\lambda_1$ ,  $\lambda_{1c}$ ,  $\lambda_{2f}$  and  $z$  as functions of the parameters of the consumption and dividend growth dynamics and of the recursive utility function defined above.

### 3.2 The Price-Consumption Ratio

We start our analysis by characterizing the vectors  $\lambda_{1c}$  and  $z$  defined in (3.8) and (3.10) that characterizes the price-consumption ratio. The characterization of this vector is the main difference between Epstein-Zin and CCAPM models. We will show below that when one has the vector  $\lambda_{1c}$ , one gets the vectors price-dividend ratio (i.e. the vector  $\lambda_1$ ) and the risk-free rate (i.e., the vector  $\lambda_{2f}$ ) as for the CCAPM. The following proposition characterizes the vectors  $\lambda_{1c}$  and  $z$ .

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<sup>4</sup>The reason is the following: the function  $g(\zeta_t)$  takes the values  $g_1$  is state 1,  $g_2$  is state 2, ...,  $g_N$  is state N; hence,

$$g(\zeta_t) = \bar{g}^\top \zeta_t \quad \text{where} \quad \bar{g} = (g_1, g_2, \dots, g_N)^\top.$$

**Proposition 3.1** *When the EIS is different from one, the components  $\lambda_{1c,i}$ ,  $i = 1, \dots, N$ , of the vector  $\lambda_{1c}$  are solution of the following equations:*

$$\lambda_{1c,i} = \delta \left( \sum_{j=1}^N p_{ij} (\lambda_{1c,j} + 1)^\theta \right)^{\frac{1}{\theta}} \exp \left( \frac{(1-\gamma)}{\theta} \mu_{c,i} + \frac{(1-\gamma)^2}{2\theta} \omega_{c,i} \right). \quad (3.14)$$

*Instead, if the EIS is equal to one, the components of the vector  $z$  are given by:*

$$z_i = \exp \left( \delta \mu_{c,i} + \frac{\delta(1-\gamma)}{2} \omega_{c,i} \right) \left( \sum_{j=1}^N p_{ij} z_j^{1-\gamma} \right)^{\frac{\delta}{1-\gamma}}. \quad (3.15)$$

These equations are solved for the  $\lambda_{1c}$  and the  $z$  vectors.<sup>5</sup>

We mentioned in the introduction that one advantage of developing closed-form formulas is to be able to assess the errors associated with approximations currently used to compute the price-consumption ratio. We consider the two main approximations that have been used to solve asset pricing models. The first approximation is the log-linearization of Campbell and Shiller (1988) for the market return, which leads to:

$$r_{M,t+1} = \ln R_{M,t+1} \approx k_0 + k_1 v_{1c}^\top \zeta_{t+1} - v_{1c}^\top \zeta_t + \Delta c_{t+1}, \quad (3.16)$$

where  $v_{1c}$  denotes the logarithm of the price-consumption ratio. The second one has been recently considered by Hansen, Heaton and Li (2005). They log-linearize the stochastic discount factor around the unitary elasticity of intertemporal substitution. This approximation has the advantage, compared to the latter one, not to log-linearize the market return around the endogenous price-consumption ratio (which is present in the parameter  $k_1$  in (3.16)). The formulas for these two approximations and a discussion of the choice of the parameters is included in Appendix A. The validity of these approximations will be studied in the empirical section.

### 3.3 The Price-Dividend Ratio

Given the price-consumption ratio derived in the previous section, one gets the following expressions for the equity price-dividend ratio.

**Proposition 3.2** *When the EIS is different from one, the components  $\lambda_{1i}$ ,  $i=1, \dots, N$ , of the vector  $\lambda_1$  characterizing the equity price-dividend ratio are given by the following formula:*

$$\lambda_{1i} = \delta \left( \frac{\delta}{\lambda_{1c,i}} \right)^{\theta-1} \exp \left( \mu_{cd,i} + \frac{\omega_{cd,i}}{2} \right) \left( \left( (\lambda_{1c} + e)^{\theta-1} \right)^\top P \right) \left[ Id - \delta A_* \left( \mu_{cd} + \frac{\omega_{cd}}{2} \right) \right]^{-1} e_i \quad (3.17)$$

<sup>5</sup>Equations (3.14) and (3.15) are highly nonlinear, respectively when  $\theta \neq 1$  and when  $\delta \neq 1$ . However, it is easy to solve these equations numerically by using numerical algorithms. We did by using the nonlinear equation solver in GAUSS.

where

$$A_*(u) = \text{Diag} \left( \left( \delta \frac{\lambda_{1c,1} + 1}{\lambda_{1c,1}} \right)^{\theta-1} \exp(u_1), \dots, \left( \delta \frac{\lambda_{1c,N} + 1}{\lambda_{1c,N}} \right)^{\theta-1} \exp(u_N) \right) P, \forall u \in \mathbb{R}^N. \quad (3.18)$$

and  $\lambda_{1c,i}$ ,  $i = 1, \dots, N$  are given by (3.14).

The vectors  $\mu_{cd}$  and  $\omega_{cd}$  are defined by:

$$\mu_{cd} = -\gamma\mu_c + \mu_d, \quad \omega_{cd} = \gamma^2\omega_c + \omega_d - 2\gamma(\rho \odot (\omega_c)^{1/2} \odot (\omega_d)^{1/2}),$$

and the vector  $e$  has all its components equal to one.

Instead, if the EIS is equal to one, the components of the vector  $\lambda_1$  are given by:

$$\lambda_i = \delta \left( \frac{1}{z_i^{1/\delta}} \right)^{1-\gamma} \exp \left( \mu_{cd,i} + \frac{\omega_{cd,i}}{2} \right) \left( (z^{1-\gamma})^\top P \right) \left[ Id - \delta A_1 \left( \mu_{cd} + \frac{\omega_{cd}}{2} \right) \right]^{-1} e_i \quad (3.19)$$

where  $z_i$ ,  $i = 1, \dots, N$  are given by (3.15) and

$$A_1(u) = \text{Diag} \left( z_1^{(1-1/\delta)(1-\gamma)} \exp(u_1), \dots, z_N^{(1-1/\delta)(1-\gamma)} \exp(u_N) \right) P, \forall u \in \mathbb{R}^N.$$

We also detail the two approximations of Campbell and Shiller (1988) and Hansen, Heaton and Li (2005) for the price dividend ratio in Appendix A and compare them to this exact expression in the empirical section.

### 3.4 The Risk-Free Bond

The following proposition characterizes the components of the vector  $\lambda_{2f}$  in (3.13).

**Proposition 3.3** *When the EIS is different from one, the components  $\lambda_{2f,i}$ ,  $i=1, \dots, N$ , of the vector  $\lambda_{2f}$  characterizing the risk-free rate are given by the following formula:*

$$\frac{1}{\lambda_{2f,i}} = \delta \exp \left( -\gamma\mu_{c,i} + \frac{1}{2}\gamma^2\omega_{c,i} \right) \sum_{j=1}^N p_{ij} \left( \delta \frac{\lambda_{1c,j} + 1}{\lambda_{1c,i}} \right)^{\theta-1}, \quad (3.20)$$

where  $\lambda_{1c,i}$ ,  $i = 1, \dots, N$  are given by (3.14).

Instead, if the EIS is equal to one, the components of the vector  $\lambda_{2f}$  are given by:

$$\frac{1}{\lambda_{2f,i}} = \delta \exp \left( -\mu_{c,i} - \frac{1}{2}(1-2\gamma)\omega_{c,i} \right). \quad (3.21)$$

Approximations for the risk-free bond are given in Appendix A.

## 4 Analytical Formulas for Statistics Reproducing Stylized Facts

In his survey on consumption-based asset pricing Campbell (2002) enumerates a number of stylized facts about the stock market and its relation to short-term interest rates and consumption growth. We report these stylized facts and others computed with a post-war data set of quarterly consumption, dividends and returns data for the US economy (1947:1 to 2002:4). The empirical predictability results for the quarterly US data from 1947 to 2002 are reported in table 1.

1. The average return on stock is high (7.43% per year).
2. The average riskless real interest rate is low (1.20% per year).
3. Real stock returns are volatile (standard deviation of 16.93% per year).
4. The real interest rate is much less volatile (standard deviation of 2.28% per year) and much of the volatility is due to short-run inflation risk. Note however that there might be regimes as shown in Garcia and Perron (1996).
5. Real consumption growth is very smooth (standard deviation of 1.33% per year).
6. Real dividend growth is extremely volatile at short horizons because of seasonality in dividend payments (annualized quarterly standard deviation of 22.50%). At longer horizons it is intermediate between the volatility of stock return and the volatility of consumption growth.
7. Quarterly real consumption growth and real dividend growth have a very weak correlation of 0.15 but the correlation increases at lower frequencies.
8. Real consumption growth and real stock returns have a quarterly correlation of 0.16. The correlation increases at 0.31 at a 1-year horizon and declines at longer horizons.
9. Quarterly real dividend growth and real stock returns have a very weak correlation of 0.11, but correlation increases dramatically at lower frequencies.
10. Real US consumption growth not well forecast by its own history or by the stock market. The first-order autocorrelation of the quarterly growth rate of real nondurables and services consumption is 0.22. The log price-dividend ratio forecasts less than 4.5% of the variation of real consumption growth at horizons of 1 to 4 years.

11. Real US dividend growth has some short-run forecastability arising from the seasonality of dividend payments (autocorrelation of -0.44). But it is not well forecast by the stock market. The log price-dividend ratio forecasts no more than 1.5% of the variation of real dividend growth at horizons of 1 to 4 years.
12. The real interest rate has some positive serial correlation; its first-order autocorrelation is 0.63. However the real interest rate is not well forecast by the stock market.
13. Excess returns of US stock over Treasury bills are highly forecastable. The log price-dividend ratio forecasts 10% of the variance of the excess return at a 1-year horizon, 19% at a 3-year horizon and 26% at a 5-year horizon. Real returns exhibit a lower predictability, also increasing with the horizon (9% at a 1-year horizon, 15% at a 3-year horizon and 22% at a 5-year horizon).

To reproduce these stylized facts one needs three main types of formulas: formulas for expected returns and unconditional moments of returns, formulas for predictability of returns and consumption and dividend growth by the dividend-price ratio, and formulas for variance ratios of returns.

#### 4.1 Formulas for Expected Returns

In order to study the predictability of the returns and excess returns, we need to connect them to the state variable  $\zeta_t$  and to the dividend growth. We define the return process,  $R_{t+1}$ , and aggregate returns over  $h$  periods,  $R_{t+1:t+h}$ , as:

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} = (\lambda_2^\top \zeta_t) (\lambda_3^\top \zeta_t) \exp(x_{d,t+1}) \quad \text{and} \quad R_{t+1:t+h} = \sum_{j=1}^h R_{t+j}, \quad (4.1)$$

with  $\lambda_3 = \lambda_1 + e$ .

We also denote excess returns by  $R_{t+1}^e$  and aggregate excess returns by  $R_{t+1:t+h}^e$  i.e.,

$$R_{t+1}^e = R_{t+1} - R_{f,t+1} \quad \text{and} \quad R_{t+1:t+h}^e = R_{t+1:t+h} - R_{f,t+1:t+h}. \quad (4.2)$$

We have

$$E[R_{t+1} | J_t] = (\lambda_2^\top \zeta_t) \exp(\mu_d^\top \zeta_t + \omega_d^\top \zeta_t / 2) \lambda_3^\top P \zeta_t = \psi^\top \zeta_t, \quad (4.3)$$

where  $\psi = (\psi_1, \dots, \psi_N)^\top$  and

$$\psi_i = \lambda_{2i} \exp(\mu_{d,i} + \omega_{d,i}/2) \lambda_3^\top P e_i, \quad i = 1, \dots, N. \quad (4.4)$$

Likewise,

$$E[R_{t+1}^e | J_t] = (\psi - \lambda_{2f})^\top \zeta_t. \quad (4.5)$$

Consequently,  $\forall j \geq 2$

$$E[R_{t+j} | J_t] = \psi^\top P^{j-1} \zeta_t \text{ and } E[R_{t+j}^e | J_t] = (\psi - \lambda_{2f})^\top P^{j-1} \zeta_t. \quad (4.6)$$

Finally,

$$E[R_{t+1:t+h} | J_t] = \psi_h^\top \zeta_t \text{ and } E[R_{t+1:t+h}^e | J_t] = (\psi_h - \lambda_{h,2f})^\top \zeta_t \quad (4.7)$$

where

$$\psi_h = \left( \sum_{j=1}^h P^{j-1} \right)^\top \psi \text{ and } \lambda_{h,2f} = \left( \sum_{j=1}^h P^{j-1} \right)^\top \lambda_{2f}. \quad (4.8)$$

## 4.2 Variance Ratios of Aggregate Returns

The formula for the variance ratio measures the autocorrelation in returns or excess returns. Cecchetti, Lam and Mark (1990) were the first to reproduce the autocorrelation in returns with a Lucas-type model where the growth rate of the endowment process (represented either by consumption, income or dividends) followed a two-state Markov-switching model in the mean. Bonomo and Garcia (1994) showed that a two-state model with one mean and two variances is closer to the data but cannot reproduce the autocorrelation in returns.

We compute

$$Ratio(h) \equiv \frac{1}{h} \frac{Var[R_{t+1:t+h}]}{Var[R_{t+1:t+1}]} \quad (4.9)$$

where

$$\begin{aligned} Var[R_{t+1:t+h}] &= h\theta_2^\top E[\zeta_t \zeta_t^\top] P^\top \theta_3 \\ &+ h(\theta_1 \odot \theta_1)^\top E[\zeta_t \zeta_t^\top] P^\top (\lambda_3 \odot \lambda_3) - h^2 (\theta_1^\top E[\zeta_t \zeta_t^\top] P^\top \lambda_3)^2 \\ &+ 2 \sum_{j=2}^h (h-j+1) \theta_1^\top E[\zeta_t \zeta_t^\top] P^\top (\lambda_3 \odot ((P^{j-2})^\top (\theta_1 \odot (P^\top \lambda_3)))) \end{aligned} \quad (4.10)$$

and

$$\theta_1 = \lambda_2 \odot (\exp(\mu_{d,1} + \omega_{d,1}/2), \dots, \exp(\mu_{d,N} + \omega_{d,N}/2))^\top, \quad (4.11)$$

$$\theta_2 = (\theta_1 \odot \theta_1 \odot (\exp(\omega_{d,1}), \dots, \exp(\omega_{d,N}))^\top) - (\theta_1 \odot \theta_1), \quad (4.12)$$

$$\theta_3 = \lambda_3 \odot \lambda_3. \quad (4.13)$$

One also gets a similar formula for the excess returns. The variance of aggregate excess returns is provided in appendix.

### 4.3 Predictability of Returns: An Analytical Evaluation

Stylized facts show a strong predictability of (excess) returns by the dividend-price ratio, which increases with the horizon. It is important to establish if this predictability measured inevitably in finite samples is reproduced in population by the postulated model. Therefore, we provide below the formulas for the population coefficients of determination of the regressions of aggregated returns over a number of periods on the price-dividend ratio.

It is common in the asset pricing literature to predict future (excess) returns by the dividend-price ratio. In doing so, one computes the regression of the aggregate returns onto the dividend-price ratio and a constant. In the following, we will use the analytical formulas derived above in order to study these predictive ability in population.

When one does the linear regression of a variable, say  $y_{t+1:t+h}$ , onto by another one, say  $x_t$ , and a constant, one gets

$$y_{t+1:t+h} = a_y(h) + b_y(h)x_t + \eta_{y,1,t+h}(h) \quad (4.14)$$

where

$$b_y = \frac{Cov(y_{t+1:t+h}, x_t)}{Var[x_t]} \quad (4.15)$$

while the corresponding population coefficient of determination denoted  $R^2(y, x, h)$  is given by:

$$R^2(y, x, h) = \frac{(Cov(y_{t+1:t+h}, x_t))^2}{Var[y_{t+1:t+h}]Var[x_t]}. \quad (4.16)$$

In order to use these formulas to characterize the predictive ability of dividend-price and consumption-price ratios for returns, one needs the variance of payoff-price ratios, covariances of payoff-price ratios with aggregate returns and variance of aggregate returns. One has:

$$Var \left[ \frac{D_t}{P_t} \right] = \lambda_2^\top Var[\zeta_t] \lambda_2 \quad \text{and} \quad Var \left[ \frac{C_t}{P_{M,t}} \right] = \lambda_{2c}^\top Var[\zeta_t] \lambda_{2c}, \quad (4.17)$$

$$Cov \left( R_{t+1:t+h}, \frac{D_t}{P_t} \right) = \psi_h^\top Var(\zeta_t) \lambda_2 \quad \text{and} \quad Cov \left( R_{t+1:t+h}, \frac{C_t}{P_{M,t}} \right) = \psi_h^\top Var(\zeta_t) \lambda_{2c}. \quad (4.18)$$

and the variance of aggregate returns is given by (4.10). Similar formulas for excess returns, consumption volatility, consumption growth and dividend growth are provided in the appendix.

## 5 Applications to Models of the Post-War US Economy

First, we analyze two models that advocate the determining role of economic uncertainty (volatility of consumption) in the formation of asset prices. We apply the derived formulas to analyze the Markov-switching model with Epstein and Zin (1989) preferences proposed by Lettau, Ludvigson and Wachter (2004). It is a direct application of our framework and we will be able to compute directly all quantities of interest analytically from their setting. Then, we investigate the properties of the Bansal and Yaron (2004) long-run risk model. We calibrate a Markov-switching model to their endowment process in order to reproduce the stylized facts that they report. Our last application is to reproduce stylized facts reported by Campbell and Cochrane (1999) with the Markov switching model and the formulas developed in the previous sections.

In all applications, many results are based on very long simulations and several statistics have been obtained by numerical techniques. Our goal is to illustrate how easy it is to compute population values for these statistics and to do it for a larger parameter set than the limited ones typically reported in the papers. This will hopefully make it easier to understand the economic intuition behind results and assess robustness to changes in the values of preference and endowment parameters. In some cases we will explore stylized facts that have not been considered in the original papers.

### 5.1 The Lettau, Ludvigson and Wachter (2005) Model

The endowment process in this paper is a constrained version of the general process (2.3), (2.4). They assume a consumption process (2.3) where the mean and the variance are governed by two different Markov chains. For the dividend process they simply assume that  $D_t = (C_t)^\lambda$ . Therefore the mean and the standard deviation of dividend growth is simply  $\lambda$  times the mean and the standard deviation of consumption growth, and the correlation parameter is one. We report in Panel A of table 2 the corresponding values of the resulting four-state Markov chain based on the estimates reported in their paper.

In their model, they assume that investors do not know the state they are in but they know the parameters of the process. Therefore at each period they update their estimate of the probability of being in a state given their current information. In other words they compute filtered probabilities. Based on the latter, they compute numerically the price-consumption and price-dividend ratios that are solutions of the Euler conditions of the equilibrium model. We have seen that given the parameters of the endowment process we could calculate the price-consumption ratio by solving a nonlinear equation and the price-dividend ratio analytically in the Epstein-Zin model. Following this procedure at



each point in time by using the filtered probabilities for the Markov chain along the way, we can reproduce easily the full trajectory of the price-dividend ratio. We intend to carry out this exercise in future versions of this paper. Instead we will assume that investors know the state and compute the various statistics corresponding to the stylized facts we presented earlier.

Since Lettau, Ludvigson and Wachter (2005) focused on the trajectory of the price-dividend ratio and its relationship with consumption volatility, they did not report the values for these statistics and the sensitivity of the various quantities to the values of preference parameters. We include a large set of preference parameters to see how the various economic and financial quantities change as a function of preference parameters.

### 5.1.1 Asset Pricing Implications in the LLW Model

We report in table 3 the values of the first two moments of the equity premium and the risk-free rate, as well as the means of the price-dividend and the price-consumption ratios and the standard deviations of the consumption-price and the dividend-price ratios. We have limited the risk aversion parameter  $\gamma$  to this range of values because for values below 15 we obtain negative prices for large  $\psi$  values and for values above 30 we start having problems solving the nonlinear system for the price-consumption ratios.

Several comments can be made. While the equity premium can be matched with a risk aversion of 25 to 30, the risk-free rate remains high. Negative prices appear with a  $\gamma$  of 15 and even at 20 convergence problems occur. The expected value of the price-dividend ratio takes very large values. At around a maximum of 11 percent, the volatility of the equity premium is low compared to the data, but the volatility of the risk-free rate matches well the actual value. A comparison with the previous two-state model is instructive. While a higher risk aversion is needed to increase the equity premium it matches better the level of the risk-free rate and the volatility of the equity premium and produces less convergence problems at similar levels of risk aversion. The key parameters to understand these differences are the mean and volatility of dividend growth. Limited at 13.5 percent in the high-volatility state (a direct result of setting  $\lambda$  to 4.5), the volatility is much lower than the 20 percent estimated with the dividend data. Moreover it falls at around 7 percent in the low volatility state. In the two-state model it remained at 16 percent. For the mean, it is the same multiple of the mean of consumption growth in low and high states. This does not seem to be coherent with the data, especially in the high mean state. This state is the most frequently visited with an overall probability of 86 percent. We will come back to these remarks later when we analyze the Bansal and Yaron (2004) model. It will be also a four-state model but the parameters of the dividend process will be based on the data.

### 5.1.2 Predictability by the Dividend Price Ratio in the LLW Model

Table 4 reports the results of predictability regressions where the predictor is the dividend price ratio and the predicted variables are, in turn, future equity returns and excess returns cumulated over one, three and five years, as well as consumption volatility, and consumption and dividend growth rates cumulated over the same horizons.

We report the  $R^2$  values of the regression of future returns on the current dividend-price ratio in Table 4. Before we compare the results with the data, it is important to emphasize that the statistics we compute in a quarterly model is the predictability of future returns at several horizons (in the table we report 1, 3 and 5 years) based on the current quarterly price-dividend ratio, that is computed with the dividend of the current quarter. In the regressions we carried out in the data and reported in table 1, the independent variable was a price-dividend ratio with dividends cumulated over a year. This adds persistence to the regressor and increases the  $R^2$  of the regression. However this difference will not affect our ability to detect the ability of a model to generate predictability. To say it in a few words, the models do not seem to produce predictability at any horizon for any parameter configuration.

It is not the case with excess returns. We also report the  $R^2$  values of the regression of future excess returns on the current dividend-price ratio in table 4. Even if is not very high, there is a non-negligible predictability, which increases with risk aversion. The fact that dividends are perfectly correlated with consumption plays certainly a role in the higher predictability for excess returns than for returns.

The other important predictability concerns the volatility of consumption, which plays a key role in explaining asset prices in both Lettau, Ludvigson and Wachter (2005) and Bansal and Yaron (2004). We also report the  $R^2$  values of the regression of future consumption volatilities on the current dividend-price ratio in table 4. As expected in this model, consumption volatility is highly predictable since the dividend-price ratio depends only on the consumption states. This is totally unrealistic. The high predictability of consumption growth rates is also a feature of the model that is at odds with the observed patterns.

### 5.1.3 Variance Ratios in the LLW Model

The last point we analyzed is the capacity of the models to produce the negative autocorrelation at long horizons. The variance ratios of returns and excess returns on the stock are reported in table 5. When  $\psi$  is greater than one, the models are able to produce variance ratios less than one, declining with the horizon, for both returns and excess returns. For excess returns there is negative autocorrelation even for values of  $\psi$  less than one, but it is

more pronounced above one. These figures are consistent with the predictability detected in the data reported in Table 1.

## 5.2 The Bansal and Yaron (2004) Model

In Appendix C, we explain how we calibrate the model for the dynamics of consumption and dividends with the Markov switching framework we detailed previously. The parameters of the corresponding Markov-switching model are reported in Table (2).

### 5.2.1 Asset pricing Implications in the BW Model

In this section, we reproduce unconditional moments of the equity premium and the risk-free rate that were considered in Bansal and Yaron (2004). The set of statistics reproduced by Bansal and Yaron (2004) is given in their table 4. We present an equivalent table generated with the analytical formulas reported in the previous sections and the parameter values of the matching MS process in Table 2. We include a larger spectrum of preference parameters than in Bansal and Yaron (2004) to better understand the variation of economic and financial quantities as a function of preference parameters. To gauge the usefulness of analytical formulas it is essential to remember that in the case of the Bansal and Yaron's model, finding these quantities means either solving the model numerically for each configuration of the preference parameters or computing these quantities by simulation. Numerical solutions take time to achieve a reasonable degree of precision. For simulations, long trajectories are needed to obtain population parameters. Determining which length is appropriate is not a trivial issue, especially when coupled with time considerations.

The upper part of table 6 is based on the value of 0.998 reported by Bansal and Yaron (2004) for the time discount parameter. We observe that the values for the first two moments of the equity premium are close to the values found by Bansal and Yaron with their model, but the average risk-free rate is higher. Several interesting observations can be made from this table. First and foremost, the table shows clearly that it is through values greater than 1 for the  $\psi$  parameter that the equity premium puzzle is solved. The expected value of the equity return is about equal (around 9%) at  $\gamma = 10$  for all values of  $\psi$ . However, the risk-free rate drops five points of percentage when  $\psi$  goes from 0.5 to 1.5. At a low risk aversion, the magnitude of the drop is less pronounced. In fact, at  $\gamma = 10$  the expected value of the price-consumption ratio decreases in a significant way. A second observation concerns the price-dividend ratio. At low values of the risk aversion parameter  $\gamma$  the expectation of the price-dividend ratio increases significantly with the elasticity of intertemporal substitution  $\psi$ , while the volatility of the price-dividend does not change much. At low values of the risk aversion parameter  $\gamma$  it is exactly the opposite.

Thanks to analytical formulas it is immediate to reproduce the same table for a slightly larger  $\delta$  of 0.999. The results are presented in the lower panel of table 6. Again several instructive conclusions can be drawn. Looking only at the moments, one does not see much difference with the previous table, except maybe for the fact that the expected risk-free rate decreases, which is an expected result. However a look at the left side of the table shows that the expected values for the price-consumption ratio and the price-dividend change drastically and take very large implausible values for certain configurations of the preference parameters.<sup>6</sup>

### 5.2.2 Predictability by the Dividend Price Ratio in the BY Model

Table (7) reports the results of predictability regressions where the predictor is the dividend price ratio and the predicted variables are, in turn, future equity returns and excess returns cumulated over one, three and five years, as well as consumption volatility, and consumption and dividend growth rates cumulated over the same horizons.

Let us start by excess returns. Bansal and Yaron (2004) computed by simulation the  $R^2$  of regressions of the cumulative excess returns from  $t$  to  $t+h$  on the dividend-price ratio at  $t$ . They found that their model with a risk aversion parameter of 10 and an elasticity of intertemporal substitution of 1.5 was able to reproduce some of the predictability observed in the data. The simulation was run with 840 observations as in their data sampling period.

We have derived analytically the  $R^2$  of the same regression in population. In Panel A of Table (7) we report the corresponding results with the same configurations of preference parameters that we selected before for asset pricing implications.

The first striking result is the total lack of predictability of excess returns by the dividend-price ratio. This is in contrast with the predictability found in Bansal and Yaron (2004). They report  $R^2$  of 5, 10 and 16 percent at horizons of 1, 3 and 5 years respectively. To identify the source of these conflicting results, we first reproduce by simulation the same statistics both for the original Bansal and Yaron model (2004) and the matching Markov-switching model we have built.<sup>7</sup> We simulate the Markov-switching model over

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<sup>6</sup>An important message of Bansal and Yaron (2004) concerns the role played by time-varying volatility in consumption, a proxy for economic uncertainty. To gauge the sensitivity of the results to time-varying volatility we recomputed the same moments by keeping the volatility constant in the Markov-switching model. The corresponding asset return moments are almost identical to the results we obtained with time-varying volatility. This result is different from the result reported in Bansal and Yaron (2004) and suggests that the action is more in the time-varying mean than in the variance. This point deserves further investigation.

<sup>7</sup>A word of caution is in order. The regression that we run has as a dependent variable the cumulative monthly returns over *yearly* periods (1, 3 and 5) and the *monthly* dividend-price ratio as an independent variable. In Bansal and Yaron (2004) it is a yearly dividend (cumulated monthly dividends over twelve months). Cumulating the dividends will certainly increase the  $R^2$  but would not change the evidence over

periods of 840 observations, the sample length in Bansal and Yaron (2004), and compute the  $R^2$  of the same regression.

The results are reported in Panels B and C of Table 7. The  $R^2$  of the finite sample regressions are not as high as the figures reported in Bansal and Yaron (2004) but there is evidence of some predictability, and it is increasing with the horizon. There is a similar level of predictability both in the original Bansal and Yaron (2004) model and the matching Markov switching, so the divergence between our analytical results and the BY findings are not due to a perverse effect of our matching procedure. There is the same level of predictability at all horizons for all configurations of parameters. Therefore, we are lead to conclude that the evidence of predictability for excess returns in Bansal and Yaron (2004) is a finite sample phenomenon.<sup>8</sup> Predictability appears in finite sample due to the presence of a very persistent variable on the right hand side<sup>9</sup> but disappears in population regressions.<sup>10</sup>

Some predictability of excess returns appears if risk aversion increases for values of  $\psi$  greater than one. It is interesting to note that for higher risk aversion the BY model behaves more like the LLW model. The volatility of the stock decreases as well as the level of the price-dividend ratio.<sup>11</sup>

In Panel A of Table (7), we also report the analytical  $R^2$  of the regression of returns on equity on the dividend-price ratio for the matching MS model. In Panels B and C, the same results are shown based on simulations. There appears to be some predictability for values of the elasticity of intertemporal substitution ( $\psi$  parameter) below one, but it disappears for values above one. This is true for all values of the risk aversion parameter  $\gamma$ , the only difference being that predictability increases with  $\gamma$  for all values of  $\psi$ . Interestingly, this result about the pivotal value of one for  $\psi$  is the opposite of what was found in the previous section for asset pricing implications. The asset return moments were better reproduced for values of  $\psi$  greater than one.

For space considerations, we do not report similar predictability regression results for the consumption price ratio as a predictor variable. The recursive utility model provides the closest theoretical measure to the consumption-wealth ratio of Lettau and Ludvigson

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the actual presence of predictability. Indeed, we have verified that this different regressor is not the source of the lack of predictability

<sup>8</sup>We confirmed this result by simulating over longer sampling periods of 2,000 months. We found that results became similar to the findings with our analytical population formulas

<sup>9</sup>cite literature on problems in predictability regressions.

<sup>10</sup>Abel (2005) also finds little or no predictability of excess returns by the dividend-price ratio in a model of preferences with a benchmark level of consumption (habit formation or consumption externalities such as keeping up or catching up with the Joneses) and i.i.d. growth rates of consumption and dividends. However Abel (2005) finds that the return on stock is predictable by the dividend-price ratio

<sup>11</sup>For space considerations, these results are not reported but are available upon request from the authors.

(2001a,b). These authors have put forward that a measure of consumption over wealth has a greater predicting power than the dividend-price ratio. Indeed, we find higher predictability for all preference parameter pairs. In particular, for  $\gamma = 10$  and  $\psi = 0.5$  the  $R^2$  for the consumption-price ratio is equal to 9.39, 14.19 and 13.53 for 1, 3 and 5 years, as opposed to 7.53, 11.24 and 10.64 for the dividend-price ratio. The remarks made above about the finite sample results for the dividend-price ratio apply equally to the consumption-price ratio. In particular, there is no predictability of excess returns by the price-consumption ratio.

Another important message found in Bansal and Yaron (2004) is the predictability of consumption volatility by the dividend-price ratio. In Table 7, we report the  $R^2$  of the regression of cumulative future consumption volatility over several horizons on the current price-dividend ratio. Results are similar to those obtained for future returns predictability. Not all preference configurations are able to produce predictable volatility. Again only low values of the elasticity of intertemporal substitution are able to generate predictability ( $\psi = 0.5$ ). There is no predictability at all for values of  $\psi$  above one. Predictability is the strongest at a one-year horizon.

Finally, we look at the predictability of the consumption and dividend growth rates at the same 1, 3 and 5-year horizons. Again the evidence of predictability is strong for both consumption and dividends, much stronger than in the data. For consumption growth, predictability is around 5% at an horizon of 5 years in the data while it is predicted at some 30% in the model. Moreover, the predictability extends to dividend growth in the model (some 20% at an horizon of 5 years), since it is moved by the same state variable than consumption, yet it is non existent in the data.

In conclusion, this long-run risk model entails too much predictability of the economic fundamentals compared to the observed characteristics in the data. Additionally, predictability is often driven by a value of the elasticity of intertemporal substitution lower than one, contrary to a required value greater than one in the Bansal and Yaron (2004) setting.

### 5.2.3 Variance Ratios

There is negative autocorrelation at long horizons in returns. Evidence is provided by the variance ratios computed at several horizons in Table 1. The variance ratios are less than one after a horizon of one year and decrease up to year 4, to finally slightly increase in year 5.

The corresponding analytical quantities are reported in Table 8 along with the simulated values from the BY model and its Markov-switching counterpart. Most of the preference

parameter combinations produce strong positive autocorrelations increasing with the horizons. Only one set,  $\gamma = 5$  and  $\psi = 1.5$  produce slight negative autocorrelation. The same results would have been visible in a simulated finite sample setting with 840 observations (see Table 8). However, predictability would have appeared stronger for the above-mentioned particular set of parameters and other candidate sets.

The evidence for excess returns is more in line with the empirical findings, with a negative autocorrelation, declining with the horizon. Again, it is the combination  $\gamma = 5$  and  $\psi = 1.5$  that produces the results closest to the data.

### 5.3 Analysis of the Validity of the Loglinear Approximations

Our analytical framework is also useful to address the important issue of approximations. We need to know to what extent the types of approximations used in Bansal and Yaron (2004) or Hansen, Heaton and Li (2005) are close to the true values and how the quality of the approximations varies in the parameter space. Therefore, we will study the effect of log-linearizing the returns on the market portfolio and on the equity. We have detailed in Appendix A the approximate solutions for the recursive utility model using the Campbell-Shiller (CS) and the Hansen-Heaton-Li (HHL) approximations.

The Campbell-Shiller approximation solves for the price-consumption and the price-dividend ratios around their means. Two important remarks are in order. First, in order to compute the price-dividend one needs to have computed the price-consumption ratio. Therefore, it involves a double approximation. The fact that the approximations of the price-payoff ratios are made around their means raised the issue of using an endogenous quantity in the process of solving the model. In most of the previous studies this issue is not addressed and preset values are used for the  $k$  coefficients in the log-linear equations. Yet, the coefficients  $k$  in these equations are functions of the mean price-payoff ratios and therefore depend on both the preference parameters and the parameters of the fundamentals.

In Table 9 we report the exact values of the  $k$  parameters -  $k_0$  and  $k_1$  for the market return approximation and  $k_{m0}$  and  $k_{m1}$  for the equity return approximation, for the values of the Bansal-Yaron Markov-switching model and several values of the preference parameters. We can see for example that for a given  $\gamma$  of 5,  $k_1$  is an increasing function of  $\psi$ , while  $k_0$  is a decreasing function of the same  $\psi$ .<sup>12</sup> In many of the previous papers the  $k_1$  parameter has been set exogenously to 0.997.

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<sup>12</sup>The formula for  $k_0$  is given by:  $k_0 = -\log(k_1) - (1 - k_1)\log(1/k_1 - 1)$  where  $k_1 = 1/(1 + \exp(\overline{(d_t - p_t)}))$ , with  $\overline{(d_t - p_t)}$  the mean log dividend-price ratio. The expressions for  $k_{m0}$  and  $k_{m1}$  are similar but for the consumption price ratio instead of the dividend-price ratio

We will see how this exogenous choice may affect the values obtained for the various unconditional moments of the returns and price-payoff ratios. We assess the difference between setting the  $k$  exogenously (in Table 10) and setting them at the values implied by the preference parameters (in Table 11). With arbitrary values, the very large values for the expected price-dividend ratio observed in Table 6 for a discount parameter equal to 0.999 (for the lower values of the risk aversion parameter) disappear and one may think that the model is acceptable for all configurations of the preference parameters. Similarly, for a value of 0.998, when the model does not converge (for the lowest value of the elasticity of intertemporal substitution), the average price-dividend ratio is finite and not too unreasonable. However, for the expected returns, the figures are more in line with the population moments in Table 6.

When we compute the statistics with the  $k$  values corresponding to the preference parameters (reported in table 9), the values obtained for all moments are close to the analytical values shown in table 6. This illustrates the fact that log-linearizations must be conducted very carefully. One major obstacle is to compute the  $k$  parameters when one does not have analytical formulas for the price-payoff ratios. Therefore, even if one does not want to model the endowment with a Markov-switching structure, the values obtained with this modeling strategy for these crucial parameters could definitely help for finding the right values of the  $k$  parameters.<sup>13</sup> With the endogenous values for the coefficients  $k$ , the CS approximation is pretty robust. When convergence problems appear around values of  $\gamma$  of 10 and small values of  $\psi$ , some moments indicate that there is no equilibrium solution. For the preference parameter values where there is convergence, the approximate solutions are relatively close to the true values, but not right on.

The second type of approximation, proposed by Hansen, Heaton and Li (2005), is done around a unitary value for the elasticity of intertemporal substitution  $\psi$ . By comparing the analytical moment values reported in Table 6 to the values in Table 12 we observe that the approximation is very precise for all configurations of parameters, even for a value of  $\psi$  of 0.2, which is far from 1. Given that this approximation does not require to solve first for the average price-dividend ratio, it appears to be the approximation of choice.

## 6 Conclusion

Equilibrium asset pricing models have become harder to solve. To reproduce resilient stylized facts, researchers have assumed that the representative investor is endowed with more sophisticated preferences. The fundamentals in the economy, consumption and dividends,

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<sup>13</sup>Recently, Bansal, Kiku and Yaron (2007) have proposed a fixed point solution method.



have also been modeled with richer dynamics. Often the time required to solve the model numerically or to simulate it to compute the statistics of interest is prohibitive. Therefore, researchers lean towards simpler models, making simplifying assumptions as a compromise between reality and feasibility.

In this paper, we have provided analytical formulas that should be of great help to assess the ability of these models to reproduce a set of stylized facts. We have chosen a flexible model for the endowment that can be applied directly to the data, as already done by several researchers, or used to match other processes that are contemplated. In terms of preferences, we have chosen the recursive framework of Epstein and Zin (1989), widely used in the asset pricing literature. We have limited our analysis to the Kreps and Porteus (1978) certainty equivalent. In future research we intend to try to find analytical formulas for other certainty equivalents in the recursive framework and other types of preferences.

## A Solving the Recursive Utility Model by Approximations

### A.1 The Price-Consumption Ratio

The vector  $v_{1c}$  in the log-linearization (3.16) is given by:

$$v_{1c,i} = (\ln \delta + k_0) + \frac{(1-\gamma)}{\theta} \mu_{c,i} + \frac{(1-\gamma)^2}{2\theta} \omega_{c,i} + \frac{1}{\theta} \ln \left[ \sum_{j=1}^N p_{ij} \exp(\theta k_1 v_{1c,j}) \right], \quad (\text{A.1})$$

Notice that although the coefficient  $k_1$  is exogenously set at a given value in many previous studies, it is an endogenous coefficient which depends on the model parameters for the economic fundamentals and preferences. The value of coefficients  $k_1$  and  $k_0$  are given by the formulas:

$$k_1 = \frac{1}{1 + \exp(-\Pi^\top l_{\lambda_c})} \quad \text{and} \quad k_0 = -\ln k_1 - (1 - k_1) \ln \left( \frac{1}{k_1} - 1 \right), \quad (\text{A.2})$$

where  $l_{\lambda_c} = (\ln \lambda_{1c,1}, \dots, \ln \lambda_{1c,N})^\top$  and  $\lambda_{1c,i}$ ,  $i = 1, \dots, N$  are given by (3.14).

Instead of the log-linearization of the market return around the endogenous price-consumption ratio, Hansen, Heaton and Li (2005) log-linearize the stochastic discount factor around the unitary elasticity of intertemporal substitution.

We consider the following notations:

$$v_t = \ln \left( \frac{V_t}{C_t} \right), \quad Dv_t^1 = \lim_{\psi \rightarrow 1} \frac{\partial v_t}{\partial (1/\psi)} = h^\top \zeta_t \quad \text{and} \quad Dm_{t,t+1}^1 = \lim_{\psi \rightarrow 1} \frac{\partial m_{t,t+1}}{\partial (1/\psi)} = \zeta_{t+1}^\top F \zeta_t, \quad (\text{A.3})$$

where  $v_t$  is the logarithm of the utility-consumption ratio and  $Dv_t^1$  is its derivative with respect to  $1/\psi$  and evaluated at  $1/\psi = 1$ ,  $m_{t,t+1}$  is the logarithm of the SDF (3.3) and  $Dm_{t,t+1}^1$  is its derivative with respect to  $1/\psi$  and evaluated at  $1/\psi = 1$ .

Hansen, Heaton and Li (2005) establish that the derivative  $Dv_t^1$  is given by the recursion:

$$Dv_t^1 = -\frac{1-\delta}{2\delta} (v_t^1)^2 + \delta E \left[ \frac{(V_{t+1}^1)^{1-\gamma}}{E \left[ (V_{t+1}^1)^{1-\gamma} \mid J_t \right]} Dv_{t+1}^1 \mid J_t \right], \quad (\text{A.4})$$

from which it follows that the components  $h_i$ ,  $i = 1, \dots, N$  of the vector  $h$  characterizing this derivative in our model are given by the equation:

$$h_i = -\frac{1-\delta}{2\delta} \frac{1}{\sum_{j=1}^N p_{ij} z_j^{1-\gamma}} \left( \left( (l_z^2)^\top \right) \odot \left( (z^{1-\gamma})^\top P \right) \right) [Id - \delta A_{**}(0)]^{-1} e_i, \quad (\text{A.5})$$

where  $z_i$ ,  $i = 1, \dots, N$  are given by (3.15),  $l_z = (\ln z_1, \dots, \ln z_N)^\top$  and:

$$A_{**}(u) = \text{Diag} \left( \exp(u_1) \left( \frac{z_1^{1-\gamma}}{\sum_{j=1}^N p_{1j} z_j^{1-\gamma}} \right), \dots, \exp(u_N) \left( \frac{z_N^{1-\gamma}}{\sum_{j=1}^N p_{Nj} z_j^{1-\gamma}} \right) \right) P, \quad \forall u \in \mathbb{R}^N. \quad (\text{A.6})$$

They also establish that the derivative  $Dm_{t,t+1}^1$  is given by the equation:

$$Dm_{t,t+1}^1 = v_{t+1}^1 - \frac{v_t^1}{\delta} + (1 - \gamma) \left( Dv_{t+1}^1 - E \left[ \frac{(V_{t+1}^1)^{1-\gamma}}{E[(V_{t+1}^1)^{1-\gamma} | J_t]} Dv_{t+1}^1 | J_t \right] \right), \quad (\text{A.7})$$

from which it follows that the elements  $f_{ij}$ ,  $1 \leq i, j \leq N$  of the matrix  $F^\top$  characterizing this derivative are given by:

$$f_{ij} = (\ln z_j + (1 - \gamma) h_j) - \frac{(\ln z_i + (1 - \gamma) h_i)}{\delta} - \frac{(1 - \gamma)(1 - \delta)}{2\delta^2} (\ln z_i)^2 \quad (\text{A.8})$$

where  $h_i$ ,  $z_i$ ,  $i = 1, \dots, N$  are given by (A.5) and (3.15) respectively.

Hansen, Heaton and Li (2005) consider the first-order Taylor expansion of the SDF (3.3) around the unitary elasticity of intertemporal substitution, that is:

$$\ln M_{t,t+1} = m_{t,t+1} \approx m_{t,t+1}^1 + \left( \frac{1}{\psi} - 1 \right) Dm_{t,t+1}^1. \quad (\text{A.9})$$

Let  $P_\psi$  denotes the matrix defined by  $P_\psi^\top = [p_{\psi,ij}]_{1 \leq i,j \leq N}$  such that:

$$p_{\psi,ij} = p_{ij} \exp \left( \left( \frac{1}{\psi} - 1 \right) f_{ij} \right). \quad (\text{A.10})$$

Given the Hansen, Heaton and Li (2005)'s approximation (A.9), the components of the vector  $\lambda_{1c}$  characterizing the market price-consumption ratio are given by:

$$\lambda_{1c,i} = \delta \left( \frac{1}{z_i^{1/\delta}} \right)^{1-\gamma} \exp \left( \mu_{cc,i} + \frac{\omega_{cc,i}}{2} \right) \left( (z^{1-\gamma})^\top P_\psi \right) \left[ Id - \delta A_\psi \left( \mu_{cc} + \frac{\omega_{cc}}{2} \right) \right]^{-1} e_i \quad (\text{A.11})$$

where the  $z_i$  are given by (3.15) and

$$A_\psi(u) = \text{Diag} \left( z_1^{(1-1/\delta)(1-\gamma)} \exp(u_1), \dots, z_N^{(1-1/\delta)(1-\gamma)} \exp(u_N) \right) P_\psi, \quad \forall u \in \mathbb{R}^N. \quad (\text{A.12})$$

## A.2 The Price-Dividend Ratio

One can also use the log-linearization method to get the price-dividend ratio. The log-linearization of the equity return is given by:

$$r_{t+1} = \ln R_{t+1} \approx k_{m0} + k_{m1} v_1^\top \zeta_{t+1} - v_1^\top \zeta_t + \Delta d_{t+1}. \quad (\text{A.13})$$

The endogenous coefficients  $k_{m1}$  and  $k_{m0}$  are given by:

$$k_{m1} = \frac{1}{1 + \exp(-\Pi^\top l_\lambda)} \quad \text{and} \quad k_{m0} = -\ln k_{m1} - (1 - k_{m1}) \ln \left( \frac{1}{k_{m1}} - 1 \right) \quad (\text{A.14})$$

where  $l_\lambda = (\ln \lambda_{11}, \dots, \ln \lambda_{1N})^\top$  and the  $\lambda_{1i}$  are given by (3.17).

In a double log-linearization for both the market and equity returns, we get:

$$v_{1i} = \theta \ln \delta + (\theta - 1) k_0 + k_{m0} - (\theta - 1) v_{1c,i} + \mu_{cd,i} + \frac{1}{2} \omega_{cd,i} + \ln \left[ \sum_{j=1}^N p_{ij} \exp((\theta - 1) k_1 v_{1c,j} + k_{m1} v_{1j}) \right]. \quad (\text{A.15})$$

where the  $v_{1c,i}$  are given by (A.1).

Alternatively, the approximation (A.9) due to Hansen, Heaton and Li (2005) leads to:

$$\lambda_{1i} = \delta \left( \frac{1}{z_i^{1/\delta}} \right)^{1-\gamma} \exp \left( \mu_{cd,i} + \frac{\omega_{cd,i}}{2} \right) \left( (z^{1-\gamma})^\top P_\psi \right) \left[ Id - \delta A_\psi \left( \mu_{cd} + \frac{\omega_{cd}}{2} \right) \right]^{-1} e_i \quad (\text{A.16})$$

where the  $z_i$  are given by (3.15) and  $A_\psi$  is defined in (A.12).

## A.3 The Risk-Free Rate

The risk-free rate with the Campbell and Shiller's log-linearization of the market return is given by:

$$\frac{1}{\lambda_{2f,i}} = \delta^\theta \exp((\theta - 1)(k_0 - v_{1c,i})) \exp \left( -\gamma \mu_{c,i} + \frac{1}{2} \gamma^2 \omega_{c,i} \right) \sum_{j=1}^N p_{ij} \exp((\theta - 1) k_1 v_{1c,j}), \quad (\text{A.17})$$

where the  $v_{1c,i}$  are given by (A.1).

Alternatively, the risk-free rate with the Hansen, Heaton and Li's Taylor expansion is given by:

$$\frac{1}{\lambda_{2f,i}} = \delta \exp \left( -\mu_{c,i} - \frac{1}{2} (1 - 2\gamma) \omega_{c,i} \right) \frac{\sum_{j=1}^N p_{\psi,ij} z_j^{1-\gamma}}{\sum_{j=1}^N p_{ij} z_j^{1-\gamma}}, \quad (\text{A.18})$$

where the  $z_i$  are given by (3.15) and the  $p_{\psi,ij}$  are defined in (A.10).

## B Additional Formulas for Statistics Reproducing Stylized facts

The variance of aggregate excess returns is given by:

$$\begin{aligned}
Var [R_{t+1:t+h}^e] &= h\theta_2^\top E [\zeta_t \zeta_t^\top] P^\top \theta_3 \\
&+ h \left( (\theta_1 \odot \theta_1)^\top E [\zeta_t \zeta_t^\top] P^\top (\lambda_3 \odot \lambda_3) - 2(\theta_1 \odot \lambda_{2f})^\top E [\zeta_t \zeta_t^\top] P^\top \lambda_3 \right) \\
&+ h(\lambda_{2f} \odot \lambda_{2f})^\top \Pi - h^2 (\theta_1^\top E [\zeta_t \zeta_t^\top] P^\top \lambda_3 - \lambda_{2f}^\top \Pi)^2 \\
&+ 2 \sum_{j=2}^h (h-j+1) q_j
\end{aligned} \tag{A.19}$$

where

$$\begin{aligned}
q_j &= \theta_1^\top E [\zeta_t \zeta_t^\top] P^\top \left( \lambda_3 \odot \left( (P^{j-2})^\top (\theta_1 \odot (P^\top \lambda_3)) \right) \right) \\
&- \theta_1^\top E [\zeta_t \zeta_t^\top] P^\top \left( \lambda_3 \odot \left( (P^{j-2})^\top \lambda_{2f} \right) \right) \\
&- \lambda_{2f}^\top E [\zeta_t \zeta_t^\top] (P^{j-1})^\top (\theta_1 \odot (P^\top \lambda_3)) + \lambda_{2f}^\top E [\zeta_t \zeta_t^\top] (P^{j-1})^\top \lambda_{2f}.
\end{aligned} \tag{A.20}$$

The covariances of aggregate excess returns with payoff-price ratios are given by:

$$Cov \left( R_{t+1:t+h}^e, \frac{D_t}{P_t} \right) = (\psi_h - \lambda_{h,2f})^\top Var[\zeta_t] \lambda_2, \tag{A.21}$$

$$Cov \left( R_{t+1:t+h}^e, \frac{C_t}{P_{M,t}} \right) = (\psi_h - \lambda_{h,2f})^\top Var[\zeta_t] \lambda_2. \tag{A.22}$$

The consumption volatility  $\sigma_{ct}^2$  defined in (2.3) equals  $\omega_c^\top \zeta_t$ . Aggregate consumption volatility, consumption and dividend growth rates over  $h$  periods are defined by:

$$\sigma_{c,t+1:t+h}^2 = \sum_{j=1}^h \sigma_{c,t+j}^2, \quad \Delta c_{t+1:t+h} = \sum_{j=1}^h \Delta c_{t+j} \quad \text{and} \quad \Delta d_{t+1:t+h} = \sum_{j=1}^h \Delta d_{t+j}.$$

The expected values are given by:

$$E [\sigma_{c,t+1:t+h}^2 | J_t] = \omega_{ch}^\top \zeta_t, \quad E [\Delta c_{t+1:t+h} | J_t] = \mu_{ch}^\top \zeta_t \quad \text{and} \quad E [\Delta d_{t+1:t+h} | J_t] = \mu_{dh}^\top \zeta_t$$

where

$$\omega_{ch} = \left( \sum_{j=1}^h P^j \right)^\top \omega_c, \quad \mu_{ch} = \left( \sum_{j=1}^h P^{j-1} \right)^\top \mu_c \quad \text{and} \quad \mu_{dh} = \left( \sum_{j=1}^h P^{j-1} \right)^\top \mu_d.$$

The variances are given by:

$$Var [\sigma_{c,t+1:t+h}^2] = \omega_c^\top Var [\zeta_{t+1:t+h}] \omega_c \tag{A.23}$$

$$Var [\Delta c_{t+1:t+h}] = \mu_c^\top Var [\zeta_{t+1:t+h}] \mu_c + h\omega_c^\top \Pi \tag{A.24}$$

$$Var [\Delta d_{t+1:t+h}] = \mu_d^\top Var [\zeta_{t+1:t+h}] \mu_d + h\omega_d^\top \Pi \tag{A.25}$$

where

$$Var [\zeta_{t+1:t+h}] = \left( hI + 2 \sum_{j=2}^h (h-j+1) P^{j-1} \right) Var [\zeta_t]. \quad (\text{A.26})$$

In addition, covariances with price-payoffs ratio are given by:

$$Cov \left( \sigma_{c,t+1:t+h}^2, \frac{D_t}{P_t} \right) = \omega_{ch}^\top Var [\zeta_t] \lambda_2 \text{ and } Cov \left( \sigma_{c,t+1:t+h}^2, \frac{C_t}{P_{M,t}} \right) = \omega_{ch}^\top Var [\zeta_t] \lambda_{2c}, \quad (\text{A.27})$$

$$Cov \left( \Delta c_{t+1:t+h}, \frac{D_t}{P_t} \right) = \mu_{ch}^\top Var [\zeta_t] \lambda_2 \text{ and } Cov \left( \Delta c_{t+1:t+h}, \frac{C_t}{P_{M,t}} \right) = \mu_{ch}^\top Var [\zeta_t] \lambda_{2c} \quad (\text{A.28})$$

$$Cov \left( \Delta d_{t+1:t+h}, \frac{D_t}{P_t} \right) = \mu_{dh}^\top Var [\zeta_t] \lambda_2 \text{ and } Cov \left( \Delta d_{t+1:t+h}, \frac{C_t}{P_{M,t}} \right) = \mu_{dh}^\top Var [\zeta_t] \lambda_{2c}. \quad (\text{A.29})$$

## C Reproducing the Bansal and Yaron (2004) Model with a Markov-Switching Model

The model of Bansal and Yaron (2004) for the endowment is:

$$x_{t+1} = (1 - \rho_x) \mu_x + \rho_x x_t + \varphi_e \sqrt{h_t} e_{t+1} \quad (\text{A.30})$$

$$h_{t+1} = (1 - \nu_1) \sigma^2 + \nu_1 h_t + \sigma_w w_{t+1} \quad (\text{A.31})$$

$$x_{c,t+1} = x_t + \sqrt{h_t} \eta_{t+1} \quad (\text{A.32})$$

$$x_{d,t+1} = \mu_{xd} + \phi (x_t - \mu_x) + \varphi_d \sqrt{h_t} u_{t+1} \quad (\text{A.33})$$

with  $e_{t+1}, w_{t+1}, \eta_{t+1}, u_{t+1} \sim N.i.i.D.(0, 1)$ .

Our goal here is to characterize a Markov Switching (MS) model described as in Section 2 that has the same features as the endowment model chosen by Bansal and Yaron (2004). The main characteristics of the later endowments are: 1) The expected means of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted  $x_t$ . 2) The conditional variances of the consumption and dividend growth rates are a linear function of the same autoregressive process of order one denoted  $h_t$ . 3) The variables  $x_{t+1}$  and  $h_{t+1}$  are independent conditionally to their past. 4) The innovations of the consumption and dividend growth rates are independent given the state variables.

In the MS case, the first characteristic of Bansal and Yaron (2004) model implies that one has to assume that the expected means of the consumption and dividend growth rates are a linear function of the same Markov chain with two states given that a two-state Markov chain is an AR(1) process. Likewise, the second one implies that the conditional variances of the consumption and dividend growth rates are a linear function of the same two-state Markov chain. The third characteristic implies that the two Markov chains should be independent. Consequently, we should assume that the Markov chain described in Section 2 has 4 states, two states for the conditional mean and two states for the conditional variance and that the transition matrix  $P$  is restricted such as conditional mean and variance are independent. Finally, the last characteristic implies that the vector  $\rho$  defined in (2.5) is equal to zero.

We would like to approximate an AR(1) process, say  $z_t$ , like  $x_t$  or  $h_t$  by a two-state Markov chain. Without loss of generality, we assume that the Markov chain  $y_t$  takes the values 0 (first state) and 1 (second state) while the transition matrix  $P_y$  is given by

$$P_y^\top = \begin{pmatrix} p_{y,11} & 1 - p_{y,11} \\ 1 - p_{y,22} & p_{y,22} \end{pmatrix}.$$

The stationary distribution is

$$\pi_{y,1} = P(y = 0) = \frac{1 - p_{y,22}}{2 - p_{y,11} - p_{y,22}}, \quad \pi_{y,2} = P(y = 1) = \frac{1 - p_{y,11}}{2 - p_{y,11} - p_{y,22}}. \quad (\text{A.34})$$

In addition, we assume that  $z_t = a + by_t$ . Without loss of generality, we assume that  $b > 0$ , that is, the second state corresponds to this high value of  $z_t$ . Our goal is to characterize the vector  $\theta = (p_{y,11}, p_{y,22}, a, b)^\top$  that matches the characteristic of the process  $z_t$ . The first characteristics that we want to match are the mean, the variance and the first order autocorrelation of the process  $z_t$  denoted  $\mu_z$ ,  $\sigma_z^2$  and  $\rho_z$  respectively. Given that the dimension of  $\theta$  is four, another restriction is needed. For instance, Mehra and Prescott (1985) assumed  $p_{y,11} = p_{y,22}$ . In contrast, we will focus on matching the kurtosis of the process  $z_t$  denoted  $\kappa_z$ . We will show below that matching the mean, variance, kurtosis and first autocorrelation does not fully identify the parameters. However, knowing the sign of the skewness of  $z_t$  (denotes  $sk_z$ ) and the other four characteristics will fully identify the vector  $\theta$ .

The moments of the AR(1) process  $z_t$  are related to those of the two-state Markov chain  $y_t$  as follows:

$$\begin{aligned} \mu_z &= a + b\mu_y = a + b\pi_{y,2} \\ \sigma_z^2 &= b^2\sigma_y^2 = b^2\pi_{y,1}\pi_{y,2} \\ sk_z &= sk_y = \left( -\frac{\pi_{y,2}}{\pi_{y,1}} + \frac{\pi_{y,1}}{\pi_{y,2}} \right) \frac{1}{\sqrt{\pi_{y,1}\pi_{y,2}}} \\ \kappa_z &= \kappa_y = \frac{\pi_{y,1}^2}{\pi_{y,2}} + \frac{\pi_{y,2}^2}{\pi_{y,1}} \\ \rho_z &= \rho_y = p_{y,11} + p_{y,22} - 1 \end{aligned}$$

The previous proposition, combined with (A.34), characterizes the moments of a Markov chain in terms of the vector  $\theta$ . As pointed out above, Mehra and Prescott (1985) assumed that  $p_{y,11} = p_{y,22}$ , which implies  $sk_z = 0$  and  $\kappa_z = 1$ . The empirical evidence reported in Cecchetti, Lam and Mark (1990) suggests that the kurtosis of the expected consumption growth is higher than one and that its skewness is negative.<sup>14</sup> We will now invert this characterization, that is, we will determine the vector  $\theta$  in terms of the moments of  $z_t$ :

The vector  $\theta$  of parameters of the two-state Markov chain that matches the AR(1)

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<sup>14</sup>Strictly speaking, the process  $x_t$  here is the expected mean of the consumption growth and not the growth. Therefore, the skewness and kurtosis of these two processes are different but connected.

process  $z_t$  is given by:

$$p_{y,11} = \frac{1 + \rho_z}{2} - \frac{1 - \rho_z}{2} \sqrt{\frac{k_z - 1}{k_z + 3}}, \quad p_{y,22} = \frac{1 + \rho_z}{2} + \frac{1 - \rho_z}{2} \sqrt{\frac{k_z - 1}{k_z + 3}} \quad \text{if } sk_z \leq 0, \quad (\text{A.35})$$

$$p_{y,11} = \frac{1 + \rho_z}{2} + \frac{1 - \rho_z}{2} \sqrt{\frac{k_z - 1}{k_z + 3}}, \quad p_{y,22} = \frac{1 + \rho_z}{2} - \frac{1 - \rho_z}{2} \sqrt{\frac{k_z - 1}{k_z + 3}} \quad \text{if } sk_z > 0, \quad (\text{A.36})$$

$$b = \frac{\sigma_z}{\sqrt{\pi_{y,1}\pi_{y,2}}}, \quad a = \mu_z - b\pi_{y,2} \quad (\text{A.37})$$

and  $\pi_{y,1}$  and  $\pi_{y,2}$  are connected to  $p_{y,11}$  and  $p_{y,22}$  through (A.34).

The mean  $\mu_x$  and the first autocorrelation  $\rho_x$  of  $x_t$  are given in (A.30). The variance, skewness and kurtosis of  $x_t$  are given by

$$\sigma_x^2 = \frac{\varphi_e^2 \sigma^2}{1 - \rho_x^2}, \quad sk_x = 0, \quad \kappa_x = 3 \frac{(1 - \rho_x^2)^2}{1 - \rho_x^4} \left( 1 + 2 \frac{\rho_x^2}{1 - \rho_x^2} \frac{\nu_1}{\sigma^2} + \frac{\sigma_w^2}{\sigma^2(1 - \nu_1^2)} \right). \quad (\text{A.38})$$

Likewise,

$$\mu_h = \sigma^2, \quad \sigma_h^2 = \frac{\sigma^2}{1 - \nu_1^2}, \quad sk_h = 0, \quad \kappa_h = 3, \quad \rho_h = \nu_1. \quad (\text{A.39})$$

Observe that the skewness of the conditional mean of consumption growth equals zero in Bansal and Yaron (2004) as in Mehra and Prescott (1985). In contrast, in order to generate a kurtosis higher than one, the Markov switching needs some skewness. Given that the skewness of consumption growth is empirically negative, we make this identification assumption, that is, we use (A.35) to identify the transition probabilities  $p_{x,11}$  and  $p_{x,22}$ .

Likewise, the skewness of the conditional variance is zero in Bansal and Yaron (2004), somewhat unrealistic given that the variance is a positive random variable. A popular variance model is the Heston (1993) model where the stationary distribution of the variance process is a Gamma distribution. Given that the skewness of a Gamma distribution is positive, we make the same assumption on  $h_t$  and therefore, use (A.36) to identify the transition probabilities  $p_{h,11}$  and  $p_{h,22}$ .

We do have now the two independent Markov chains that generate the expected mean and variance of consumption growth. Putting together these two processes leads to a four-state Markov chain (low mean and low variance, low mean and high variance, high mean and low variance, high mean and high variance) whose transition probability matrix is given by

$$P^\top = \begin{bmatrix} p_{x,11}p_{h,11} & p_{x,11}p_{h,12} & p_{x,12}p_{h,11} & p_{x,12}p_{h,12} \\ p_{x,11}p_{h,21} & p_{x,11}p_{h,22} & p_{x,12}p_{h,21} & p_{x,12}p_{h,22} \\ p_{x,21}p_{h,11} & p_{x,21}p_{h,12} & p_{x,22}p_{h,11} & p_{x,22}p_{h,12} \\ p_{x,21}p_{h,21} & p_{x,21}p_{h,22} & p_{x,22}p_{h,21} & p_{x,22}p_{h,22} \end{bmatrix} \quad (\text{A.40})$$

where  $p_{\cdot,12} = 1 - p_{\cdot,11}$  and  $p_{\cdot,21} = 1 - p_{\cdot,22}$ , while the vectors  $\mu_c$ ,  $\omega_c$ ,  $\mu_d$ , and  $\omega_d$  defined in (2.3) and (2.4) are given by

$$\begin{aligned} \mu_c &= (a_c, a_c, a_c + b_c, a_c + b_c)^\top \\ \omega_c &= (a_h, a_h + b_h, a_h, a_h + b_h)^\top \\ \mu_d &= (\mu_{xd} - \phi\mu_x)e + \phi\mu_c \\ \omega_d &= \varphi_d^2 \omega_c. \end{aligned} \quad (\text{A.41})$$



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Table 1: **Predictability of Returns and Growth Rates: Data.**

This table shows estimates of slope coefficients, and R-squared of regressions  $y_{t+1:t+h} = a_y(h) + b_y(h) \frac{D_t}{P_t} + \eta_{y,t+h}(h)$ , where the variable  $y$  is return, excess return, consumption growth rate or dividend growth rate. Standard errors are Newey and West (1987) corrected using 10 lags. Lines 6 and 11 show variance ratios of aggregate returns and aggregate excess returns respectively. The horizon  $h$  is quarterly in regressions and converted into annual in the table. Estimates and standard deviations of slope coefficients are multiplied by  $10^{-4}$  in the table.

h	1	2	3	4	5
<b>Returns</b>					
Estimate	0.1416	0.2415	0.3027	0.3747	0.5128
Std. Dev.	0.0502	0.0930	0.1166	0.1277	0.1498
R-squared	9.0192	13.5480	15.1060	17.5200	22.3720
Var. Ratio	1.0271	0.9623	0.8660	0.8209	0.9199
<b>Excess Returns</b>					
Estimate	0.1527	0.2617	0.3247	0.3875	0.5126
Std. Dev.	0.0458	0.0858	0.1066	0.1156	0.1354
R-squared	10.9800	17.1750	19.5180	21.8600	26.2010
Var. Ratio	1.0028	0.9105	0.7880	0.7189	0.8017
<b>Consumption Growth</b>					
Estimate	-0.0047	-0.0058	-0.0117	-0.0163	-0.0214
Std. Dev.	0.0041	0.0073	0.0098	0.0117	0.0136
R-squared	1.6140	1.0546	2.6146	3.6245	4.6223
<b>Dividend Growth</b>					
Estimate	0.0045	0.0184	0.0233	0.0370	0.0623
Std. Dev.	0.0176	0.0337	0.0431	0.0503	0.0536
R-squared	0.0488	0.4107	0.4435	0.9214	2.3147

Table 2: **Parameters of Markov-Switching Models.**

In this table, we report in Panel A the parameters of the four-state quarterly Markov-Switching Model based on estimates of Lettau, Ludvigson and Wachter (2006). In Panel B, we report the parameters of the four-state monthly Markov-switching model that matches the long-run risk model of Bansal and Yaron (2004) model.  $\mu_c$  and  $\mu_d$  are conditional means of consumption and dividend,  $\omega_c$  and  $\omega_d$  are conditional variances of consumption and dividend.  $P^\top$  is the transition matrix across different regimes and  $\Pi$  is the vector of unconditional probabilities of regimes. Means and standard deviations are in percent.

	<b>A. Lettau et al. (2006)</b>				<b>B. Bansal and Yaron (2004)</b>			
	State 1	State 2	State 3	State 4	State 1	State 2	State 3	State 4
$\mu_c^\top$	0.62	0.62	-0.32	-0.32	-0.18	-0.18	0.24	0.24
$\mu_d^\top$	2.80	2.80	-1.45	-1.45	-0.84	-0.84	0.41	0.41
$(\omega_c^\top)^{1/2}$	0.75	0.40	0.75	0.40	0.73	0.94	0.73	0.94
$(\omega_d^\top)^{1/2}$	3.36	1.82	3.36	1.82	3.29	4.23	3.29	4.23
	$P^\top$				$P^\top$			
State 1	0.960	0.006	0.034	0.000	0.981	0.003	0.017	0.000
State 2	0.009	0.957	0.000	0.034	0.010	0.973	0.000	0.017
State 3	0.205	0.001	0.789	0.005	0.004	0.000	0.993	0.003
State 4	0.002	0.204	0.007	0.787	0.000	0.004	0.010	0.985
$\Pi^\top$	0.515	0.343	0.085	0.057	0.162	0.043	0.627	0.168

Table 3: **Asset Pricing Implications: LLW.**

The entries are model population values of asset prices. Price-consumption and price-dividend ratios are exact values under no approximation. The input parameters for the quarterly model are given in Table 2. The expressions  $E[R^e]$  and  $E[R_f]$  are respectively the annualized equity premium and mean risk-free rate. The expressions  $\sigma(R)$ ,  $\sigma(R_f)$ ,  $\sigma\left(\frac{C}{P_M}\right)$  and  $\sigma\left(\frac{D}{P}\right)$  are respectively the annualized volatilities of equity return, risk-free rate, consumption-price ratio and dividend-price ratio. The quarterly subjective factor of discount is set to 0.9925.

$\gamma$	$\psi$	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
15	0.5	2.74	6.93	8.42	1.27	21.77	243.31	0.036	0.013
15	0.7	3.21	5.70	9.38	0.92	27.08	-243.94	0.013	0.016
15	1.3	3.83	4.26	10.69	0.53	37.54	-75.01	0.005	0.067
15	1.5	3.94	4.03	10.91	0.47	39.92	-67.84	0.007	0.077
20	0.5	4.11	6.99	8.69	1.26	22.73	55.92	0.038	0.062
20	0.7	4.81	5.69	9.69	0.92	27.72	90.73	0.014	0.046
20	1.3	5.70	4.14	11.00	0.53	36.89	296.39	0.006	0.017
20	1.5	5.85	3.90	11.22	0.47	38.87	451.93	0.008	0.012
25	0.5	5.57	7.11	8.71	1.28	23.93	30.16	0.038	0.116
25	0.7	6.50	5.69	9.69	0.93	28.47	36.30	0.014	0.115
25	1.3	7.68	4.01	10.96	0.53	36.20	46.92	0.006	0.106
25	1.5	7.87	3.75	11.17	0.47	37.77	49.10	0.009	0.104
30	0.5	6.92	7.27	8.49	1.32	25.38	20.77	0.038	0.163
30	0.7	8.09	5.72	9.40	0.95	29.32	22.95	0.015	0.174
30	1.3	9.54	3.87	10.60	0.53	35.49	25.99	0.007	0.182
30	1.5	9.78	3.58	10.80	0.46	36.67	26.53	0.009	0.183

Table 4: **Predictability by the Dividend-Price Ratio: LLW.**

This table shows the R-squared of the regression  $y_{t+1:t+h} = a_2(h) + b_2(h) \frac{D_t}{P_t} + \eta_{2,t+h}(h)$ , where  $y$  is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon  $h$  is quarterly in the regression and converted into annual in the table. The quarterly subjective factor of discount is set to 0.9925.

$\gamma$	$\psi$	$h$														
		Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
15	0.5	0.38	0.08	0.00	1.15	1.64	1.80	41.99	38.83	35.95	16.20	8.38	4.93	16.20	8.38	4.93
15	0.7	0.00	0.04	0.11	1.28	1.73	1.84	43.30	40.05	37.07	15.77	8.16	4.80	15.77	8.16	4.80
15	1.3	0.34	0.52	0.61	1.38	1.81	1.87	46.92	43.39	40.17	14.63	7.57	4.45	14.63	7.57	4.45
15	1.5	0.44	0.64	0.71	1.39	1.82	1.87	47.67	44.09	40.82	14.39	7.44	4.38	14.39	7.44	4.38
20	0.5	0.10	0.00	0.05	1.81	2.72	3.08	45.26	41.86	38.75	15.18	7.85	4.62	15.18	7.85	4.62
20	0.7	0.06	0.23	0.38	1.97	2.78	3.02	44.33	41.00	37.96	15.45	7.99	4.71	15.45	7.99	4.71
20	1.3	0.70	1.01	1.12	2.11	2.84	2.97	44.60	41.25	38.19	15.34	7.94	4.67	15.34	7.94	4.67
20	1.5	0.84	1.17	1.27	2.13	2.85	2.96	44.75	41.38	38.31	15.29	7.91	4.66	15.29	7.91	4.66
25	0.5	0.04	0.02	0.15	2.27	3.63	4.23	46.92	43.39	40.17	14.67	7.59	4.47	14.67	7.59	4.47
25	0.7	0.12	0.38	0.59	2.39	3.56	3.98	44.36	41.02	37.98	15.45	7.99	4.70	15.45	7.99	4.70
25	1.3	0.88	1.31	1.48	2.52	3.52	3.76	42.56	39.36	36.43	15.98	8.27	4.87	15.98	8.27	4.87
25	1.5	1.05	1.50	1.65	2.53	3.51	3.73	42.36	39.17	36.26	16.04	8.30	4.88	16.04	8.30	4.88
30	0.5	0.07	0.03	0.20	2.40	4.19	5.13	49.49	45.77	42.37	13.88	7.18	4.23	13.88	7.18	4.23
30	0.7	0.08	0.37	0.65	2.41	3.89	4.57	45.71	42.27	39.13	15.03	7.78	4.58	15.03	7.78	4.58
30	1.3	0.78	1.29	1.55	2.43	3.63	4.06	42.60	39.40	36.47	15.98	8.27	4.87	15.98	8.27	4.87
30	1.5	0.94	1.48	1.73	2.43	3.59	3.99	42.21	39.04	36.14	16.09	8.33	4.90	16.09	8.33	4.90

Table 5: **Variance Ratios of Aggregate Returns: LLW**

This table shows the variance ratios  $\frac{Var(R_{t+1:t+h})}{hVar(R_{t+1})}$  and  $\frac{Var(R_{t+1:t+h}^e)}{hVar(R_{t+1}^e)}$ , where the horizon  $h$  is quarterly and converted into annual in the table. The quarterly subjective factor of discount is set to 0.9925.

		$h$									
$\gamma$	$\psi$	Returns					Excess				
		1	2	3	4	5	1	2	3	4	5
15	0.5	1.118	1.206	1.255	1.284	1.303	0.937	0.890	0.863	0.847	0.837
15	0.7	1.035	1.061	1.075	1.083	1.088	0.921	0.861	0.827	0.807	0.793
15	1.3	0.960	0.928	0.911	0.899	0.891	0.905	0.833	0.792	0.767	0.750
15	1.5	0.950	0.911	0.889	0.875	0.866	0.903	0.829	0.787	0.761	0.744
20	0.5	1.090	1.157	1.195	1.217	1.231	0.923	0.865	0.833	0.814	0.802
20	0.7	1.010	1.018	1.022	1.023	1.024	0.905	0.834	0.794	0.770	0.754
20	1.3	0.939	0.892	0.865	0.849	0.838	0.889	0.804	0.756	0.727	0.709
20	1.5	0.929	0.876	0.845	0.826	0.814	0.886	0.800	0.751	0.722	0.702
25	0.5	1.086	1.150	1.186	1.207	1.221	0.921	0.863	0.831	0.813	0.802
25	0.7	1.007	1.012	1.014	1.015	1.016	0.903	0.831	0.791	0.767	0.752
25	1.3	0.935	0.886	0.858	0.841	0.830	0.886	0.800	0.752	0.723	0.705
25	1.5	0.926	0.869	0.838	0.818	0.805	0.884	0.796	0.747	0.718	0.698
30	0.5	1.107	1.187	1.232	1.259	1.277	0.933	0.884	0.859	0.845	0.838
30	0.7	1.025	1.043	1.054	1.060	1.064	0.915	0.853	0.819	0.799	0.787
30	1.3	0.949	0.911	0.889	0.876	0.868	0.899	0.823	0.781	0.756	0.740
30	1.5	0.939	0.893	0.867	0.852	0.842	0.897	0.819	0.776	0.750	0.734



Table 6: **Asset Pricing Implications: BY.**

The entries are model population values of asset prices. Price-consumption and price-dividend ratios are exact values under no approximation. The input parameters for the monthly model are given in Table 2. The expressions  $E[R^e]$  and  $E[R_f]$  are respectively the annualized equity premium and mean risk-free rate. The expressions  $\sigma(R)$ ,  $\sigma(R_f)$ ,  $\sigma\left(\frac{C}{P_M}\right)$  and  $\sigma\left(\frac{D}{P}\right)$  are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio.

$\gamma$	$\psi$	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
$\delta = 0.998$									
2.5	1	0.92	4.06	16.72	0.58	41.58	56.09	0.000	0.091
2.5	1.5	1.37	3.38	18.47	0.40	52.61	77.52	0.015	0.081
5	0.5	1.05	6.11	13.68	1.14	29.89	22.77	0.074	0.115
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.5	3.78	3.02	18.96	0.40	47.51	31.44	0.018	0.210
7.5	0.2	-1.66	12.11	16.60	2.89	30.50	14.00	0.268	0.279
7.5	0.5	1.82	6.60	13.26	1.21	37.51	17.51	0.060	0.122
7.5	1	4.63	3.70	16.05	0.59	41.58	19.39	0.000	0.239
7.5	1.5	5.79	2.58	17.36	0.36	43.19	20.11	0.019	0.276
10	0.2	-4.85	15.14	18.08	3.24	INF	14.88	0.000	0.307
10	0.5	1.86	7.26	12.73	1.34	48.86	15.42	0.045	0.089
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.5	7.12	2.12	15.73	0.32	40.11	16.36	0.019	0.268
$\delta = 0.999$									
2.5	1	1.02	2.86	17.17	0.58	83.25	156.54	0.000	0.035
2.5	1.5	1.53	2.16	19.12	0.40	141.35	589.05	0.006	0.011
5	0.5	1.17	4.97	13.80	1.14	48.38	29.78	0.048	0.092
5	1	3.24	2.68	17.69	0.58	83.25	39.36	0.000	0.147
5	1.5	4.19	1.77	19.51	0.40	107.21	43.66	0.008	0.159
7.5	0.2	-1.87	11.18	17.00	2.90	63.01	16.77	0.136	0.244
7.5	0.5	1.92	5.57	13.24	1.22	75.51	20.89	0.031	0.102
7.5	1	4.98	2.49	16.11	0.58	83.25	23.17	0.000	0.202
7.5	1.5	6.25	1.30	17.48	0.36	86.31	24.06	0.010	0.234
10	0.2	-4.73	13.38	18.30	3.18	INF	19.68	0.000	0.237
10	0.5	1.81	6.32	12.67	1.37	147.73	18.15	0.015	0.071
10	1	5.95	2.31	14.65	0.59	83.25	18.78	0.000	0.188
10	1.5	7.53	0.82	15.70	0.31	74.61	19.05	0.011	0.229

Table 7: Predictability by the Dividend-Price Ratio: BY.

This table shows the R-squared of the regression  $y_{t+1:t+h} = a_2(h) + b_2(h) \frac{D_t}{P_t} + \eta_{2,t+h}(h)$ , where  $y$  is return, excess return, consumption volatility, consumption growth or dividend growth. The horizon  $h$  is monthly in the regression and converted into annual in the table. Panel A shows population values implied by the MS model. Values in Panels B and C are mean across simulations. The monthly subjective factor of discount is set to 0.999.

$\gamma$	$\psi$	$h$														
		Returns			Excess			Volatility			Consumption			Dividend		
		1	3	5	1	3	5	1	3	5	1	3	5	1	3	5
A. MS, population																
5	0.5	4.16	6.62	6.49	0.12	0.24	0.26	2.27	1.86	1.54	28.62	35.16	30.59	15.68	22.85	21.40
7.5	0.5	5.73	8.88	8.58	0.04	0.07	0.08	4.97	4.07	3.38	27.73	34.06	29.64	15.19	22.14	20.73
7.5	1.5	0.04	0.07	0.08	0.18	0.36	0.40	0.48	0.39	0.32	29.22	35.89	31.23	16.01	23.32	21.84
10	0.5	7.53	11.24	10.64	0.01	0.01	0.01	18.21	14.94	12.39	23.36	28.69	24.97	12.80	18.65	17.46
10	1.5	0.70	1.22	1.27	0.07	0.13	0.14	0.91	0.75	0.62	29.07	35.71	31.07	15.93	23.21	21.73
B. BY, simulation, 1000 samples of 840 observations																
5.0	0.5	6.06	9.36	9.42	1.38	3.48	5.04	2.84	3.40	4.40	26.64	31.19	25.94	14.88	21.05	19.29
7.5	0.5	5.98	9.30	9.42	1.36	3.47	5.04	3.54	3.52	3.97	26.39	30.91	25.78	14.76	20.94	19.26
7.5	1.5	1.54	3.56	4.90	1.57	3.93	5.57	3.95	4.18	4.08	25.85	30.29	25.32	14.47	20.55	18.93
10.0	0.5	5.65	8.89	9.13	1.35	3.49	5.10	5.85	4.88	4.83	25.40	29.81	24.99	14.23	20.30	18.80
10.0	1.5	1.52	3.57	4.94	1.59	4.01	5.69	5.43	4.76	4.69	25.29	29.67	24.87	14.17	20.19	18.67
C. MS, simulation, 1000 samples of 840 observations																
5.0	0.5	4.73	7.93	8.15	2.02	4.52	5.77	6.11	5.89	6.34	23.84	27.85	23.20	13.79	19.48	18.00
7.5	0.5	5.86	9.44	9.45	1.72	4.00	5.23	8.34	7.62	7.61	23.29	27.25	22.79	13.46	19.09	17.68
7.5	1.5	2.89	5.85	6.89	3.17	6.53	7.87	4.25	4.55	4.84	24.19	28.22	23.44	14.02	19.73	18.21
10.0	0.5	7.58	11.63	11.38	1.49	3.62	4.91	21.07	17.19	14.16	20.52	24.22	20.59	11.88	17.14	16.11
10.0	1.5	2.74	5.42	6.21	2.39	5.03	6.07	4.74	4.84	5.42	24.11	28.13	23.38	13.96	19.67	18.15

Table 8: **Variance Ratios of Aggregate Returns: BY.**

This table shows the variance ratios  $\frac{Var(R_{t+1:t+h})}{hVar(R_{t+1})}$ , where the horizon  $h$  is monthly and converted into annual in the table. Panel A shows population values implied by the MS model. Values in Panels B and C are mean across 1000 simulations of 840 observations. The monthly subjective factor of discount is set to 0.999.

$\gamma$	$\psi$	$h$					
		Returns			Excess		
		1	3	5	1	3	5
A. MS, population							
5	0.5	1.12	1.32	1.47	0.99	0.97	0.96
5	1.5	0.99	0.97	0.96	0.96	0.90	0.86
7.5	0.5	1.14	1.38	1.55	1.00	0.99	0.98
7.5	1.5	1.01	1.03	1.04	0.98	0.95	0.93
10	0.5	1.17	1.47	1.69	1.00	1.00	1.01
10	1.5	1.04	1.12	1.17	1.01	1.03	1.05
B. BY, simulation							
5	0.5	1.12	1.31	1.43	0.99	0.96	0.92
5	1.5	1.03	1.05	1.06	0.99	0.96	0.93
7.5	0.5	1.12	1.31	1.42	0.99	0.96	0.92
7.5	1.5	1.03	1.06	1.06	0.99	0.96	0.93
10	0.5	1.12	1.30	1.42	0.99	0.96	0.92
10	1.5	1.03	1.06	1.06	1.00	0.97	0.93
C. MS, simulation							
5	0.5	1.09	1.24	1.32	0.98	0.94	0.90
5	1.5	0.98	0.95	0.92	0.96	0.89	0.83
7.5	0.5	1.11	1.29	1.39	0.98	0.95	0.91
7.5	1.5	1.00	1.00	0.98	0.98	0.93	0.89
10	0.5	1.14	1.36	1.50	0.99	0.96	0.92
10	1.5	1.03	1.07	1.09	1.00	1.00	0.98

Table 9: **Endogenous Coefficients of the CS Log-Linearization.**

The entries are model implied coefficients of the Campbell and Shiller (1988)'s log-linearization. Price-consumption and price-dividend ratios are exact values under no approximation. The input parameters for the monthly model are given in Table 2.

$\gamma$	$\psi$	$\delta = 0.998$				$\delta = 0.999$			
		$k_1$	$k_0$	$k_{m1}$	$k_{m0}$	$k_1$	$k_0$	$k_{m1}$	$k_{m0}$
2.5	1	0.9980	0.0144	0.9985	0.0113	0.9990	0.0079	0.9995	0.0046
2.5	1.5	0.9984	0.0118	0.9989	0.0085	0.9994	0.0050	0.9999	0.0014
5	0.5	0.9972	0.0192	0.9963	0.0242	0.9983	0.0127	0.9972	0.0193
5	1	0.9980	0.0144	0.9971	0.0200	0.9990	0.0079	0.9979	0.0153
5	1.5	0.9983	0.0129	0.9973	0.0186	0.9992	0.0063	0.9981	0.0141
7.5	0.2	0.9971	0.0198	0.9940	0.0366	0.9986	0.0107	0.9950	0.0315
7.5	0.5	0.9978	0.0158	0.9953	0.0301	0.9989	0.0086	0.9960	0.0260
7.5	1	0.9980	0.0144	0.9957	0.0278	0.9990	0.0079	0.9964	0.0240
7.5	1.5	0.9981	0.0140	0.9958	0.0271	0.9990	0.0077	0.9965	0.0233
10	0.2	1.0000	0.0000	0.9943	0.0349	1.0000	0.0000	0.9957	0.0277
10	0.5	0.9983	0.0126	0.9946	0.0335	0.9994	0.0048	0.9954	0.0292
10	1	0.9980	0.0144	0.9948	0.0325	0.9990	0.0079	0.9956	0.0285
10	1.5	0.9979	0.0149	0.9949	0.0321	0.9989	0.0087	0.9956	0.0282

Table 10: **Asset Pricing Implications: BY + CS Log-Linearization with Exogenous Coefficients.**

The entries are model population values of asset prices. Price-consumption and price-dividend ratios are exact values under Campbell and Shiller (1988)'s log-linearization with exogenous coefficients  $k_1$  and  $k_{m1}$  both equal to 0.997. The input parameters for the monthly model are given in Table 2. The expressions  $E[R^e]$  and  $E[R_f]$  are respectively the annualized equity premium and mean risk-free rate. The expressions  $\sigma(R)$ ,  $\sigma(R_f)$ ,  $\sigma\left(\frac{C}{P_M}\right)$  and  $\sigma\left(\frac{D}{P}\right)$  are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio.

$\gamma$	$\psi$	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$	
$\delta = 0.998$										
2.5	1.5	1.82	3.39	17.62	0.40	44.61	53.44	0.016	0.107	
	5	1.19	6.08	13.79	1.14	29.50	22.29	0.074	0.123	
	5	1.5	3.63	3.06	19.00	0.40	42.40	32.57	0.018	0.203
7.5	0.2	0.34	11.97	17.53	2.89	26.60	11.38	0.299	0.379	
7.5	0.5	2.71	6.47	13.52	1.20	34.83	15.56	0.063	0.157	
7.5	1.5	6.45	2.65	18.40	0.37	40.12	18.18	0.020	0.343	
10	0.2	-3.21	13.75	18.32	3.10	47.80	14.46	0.159	0.323	
10	0.5	3.62	6.96	12.94	1.30	40.37	12.64	0.052	0.135	
10	1.5	9.41	2.21	16.69	0.33	38.19	12.09	0.020	0.419	
$\delta = 0.999$										
2.5	1.5	2.48	2.19	17.62	0.40	62.29	74.61	0.012	0.077	
	5	1.11	4.88	13.78	1.14	41.19	31.13	0.053	0.088	
	5	1.5	3.93	1.85	18.99	0.40	59.21	45.48	0.013	0.146
7.5	0.2	-1.01	10.75	17.49	2.88	37.15	15.89	0.214	0.272	
7.5	0.5	2.08	5.26	13.50	1.20	48.63	21.73	0.045	0.112	
7.5	1.5	6.04	1.44	18.39	0.37	56.02	25.38	0.014	0.245	
10	0.2	-4.02	12.53	18.29	3.09	66.74	20.19	0.114	0.232	
10	0.5	2.57	5.75	12.92	1.30	56.37	17.65	0.037	0.097	
10	1.5	8.21	1.00	16.67	0.33	53.32	16.90	0.014	0.300	

Table 11: **Asset Pricing Implications: BY + CS Log-Linearization with Endogenous Coefficients.**

The entries are model population values of asset prices. Price-consumption and price-dividend ratios are exact values under Campbell and Shiller (1988)'s log-linearization with endogenous coefficients  $k_1$  and  $k_{m1}$  given in Table 9. The input parameters for the monthly model are given in Table 2. The expressions  $E[R^e]$  and  $E[R_f]$  are respectively the annualized equity premium and mean risk-free rate. The expressions  $\sigma(R)$ ,  $\sigma(R_f)$ ,  $\sigma\left(\frac{C}{P_M}\right)$  and  $\sigma\left(\frac{D}{P}\right)$  are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio.

$\gamma$	$\psi$	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$	
$\delta = 0.998$										
2.5	1.5	1.41	3.38	18.55	0.40	52.58	76.13	0.015	0.083	
	5	1.07	6.10	13.70	1.14	29.69	22.74	0.074	0.116	
	5	1.5	3.94	3.02	19.23	0.40	47.47	30.39	0.018	0.223
7.5	0.2	-1.50	11.99	16.38	2.89	26.87	13.83	0.297	0.275	
7.5	0.5	1.85	6.58	13.29	1.21	37.15	17.49	0.061	0.124	
7.5	1.5	6.02	2.58	17.60	0.36	43.14	19.39	0.019	0.294	
10	0.2	5.22	21.20	58.89	6.16	INF	8.55	0.000	1.573	
10	0.5	1.88	7.24	12.74	1.34	48.37	15.41	0.045	0.090	
10	1.5	7.32	2.12	15.85	0.32	40.07	15.90	0.019	0.281	
$\delta = 0.999$										
2.5	1.5	1.54	2.16	19.13	0.40	141.27	575.41	0.006	0.012	
	5	1.18	4.96	13.82	1.14	48.00	29.72	0.048	0.093	
	5	1.5	4.32	1.77	19.73	0.40	107.09	42.04	0.008	0.169
7.5	0.2	-1.73	11.11	16.79	2.90	54.33	16.47	0.156	0.243	
7.5	0.5	1.95	5.55	13.26	1.22	74.69	20.83	0.032	0.103	
7.5	1.5	6.46	1.30	17.70	0.35	86.20	23.14	0.010	0.250	
10	0.2	5.39	18.41	71.99	6.18	INF	13.86	0.000	1.119	
10	0.5	1.83	6.31	12.68	1.37	146.07	18.12	0.015	0.072	
10	1.5	7.71	0.82	15.80	0.31	74.52	18.50	0.011	0.240	

Table 12: **Asset Pricing Implications: BY + HHL Taylor Expansion.**

The entries are model population values of asset prices. Price-consumption and price-dividend ratios are exact values under Hansen, Heaton and Li (2006)'s Taylor expansion of the stochastic discount factor around the unitary elasticity of intertemporal substitution. The input parameters for the monthly model are given in Table 2. The expressions  $E[R^e]$  and  $E[R_f]$  are respectively the annualized equity premium and mean risk-free rate. The expressions  $\sigma(R)$ ,  $\sigma(R_f)$ ,  $\sigma\left(\frac{C}{P_M}\right)$  and  $\sigma\left(\frac{D}{P}\right)$  are respectively the annualized volatilities of market return, risk-free rate, consumption-price ratio and dividend-price ratio.

$\gamma$	$\psi$	$E[R^e]$	$E[R_f]$	$\sigma(R)$	$\sigma(R_f)$	$E\left[\frac{P_M}{C}\right]$	$E\left[\frac{P}{D}\right]$	$\sigma\left(\frac{C}{P_M}\right)$	$\sigma\left(\frac{D}{P}\right)$
$\delta = 0.998$									
2.5	1	0.92	4.06	16.72	0.58	41.58	56.09	0.000	0.091
2.5	1.5	1.37	3.38	18.47	0.40	52.61	77.78	0.015	0.081
5	0.5	1.04	6.08	13.68	1.14	29.93	22.95	0.074	0.114
5	1	2.92	3.88	17.27	0.58	41.58	28.82	0.000	0.191
5	1.5	3.78	3.02	18.96	0.40	47.52	31.47	0.018	0.210
7.5	0.2	-1.60	11.94	16.57	2.88	30.56	14.19	0.267	0.274
7.5	0.5	1.82	6.60	13.26	1.21	37.51	17.52	0.060	0.122
7.5	1	4.63	3.70	16.05	0.59	41.58	19.39	0.000	0.239
7.5	1.5	5.78	2.58	17.36	0.36	43.19	20.12	0.019	0.276
10	0.2	-4.96	15.26	18.07	3.25	-764.53	14.87	0.011	0.307
10	0.5	1.86	7.25	12.73	1.34	48.91	15.42	0.045	0.089
10	1	5.65	3.52	14.70	0.59	41.58	16.09	0.000	0.222
10	1.5	7.12	2.12	15.73	0.32	40.12	16.36	0.019	0.268
$\delta = 0.999$									
2.5	1	1.02	2.86	17.17	0.58	83.25	156.54	0.000	0.035
2.5	1.5	1.52	2.17	19.11	0.40	141.38	608.57	0.006	0.011
5	0.5	1.15	4.94	13.80	1.14	48.53	30.13	0.048	0.091
5	1	3.24	2.68	17.69	0.58	83.25	39.36	0.000	0.147
5	1.5	4.19	1.77	19.51	0.40	107.26	43.73	0.008	0.159
7.5	0.2	-1.84	11.10	16.98	2.90	63.06	16.90	0.136	0.242
7.5	0.5	1.92	5.56	13.24	1.22	75.51	20.90	0.031	0.102
7.5	1	4.98	2.49	16.11	0.58	83.25	23.17	0.000	0.202
7.5	1.5	6.25	1.30	17.48	0.36	86.31	24.06	0.010	0.234
10	0.2	-5.75	14.87	18.78	3.34	-46.18	18.35	0.188	0.264
10	0.5	1.82	6.31	12.68	1.37	148.34	18.15	0.015	0.071
10	1	5.95	2.31	14.65	0.59	83.25	18.78	0.000	0.188
10	1.5	7.53	0.82	15.70	0.31	74.63	19.05	0.011	0.229