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Modeling Market Downside Volatility*

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Abstract. We propose a new methodology for modeling and estimating time-varying downside risk and upside uncertainty in equity returns and for assessment of risk–return trade-off in financial markets. Using the salient features of the binormal distribution, we explicitly relate downside risk and upside uncertainty to conditional heteroskedasticity and asymmetry through binormal GARCH (BiN-GARCH) model. Based on S&P 500 and international index returns, we find strong empirical support for existence of significant relative downside risk, and robust positive relationship between relative downside risk and conditional mode.

Keywords: Binormal distribution, Downside risk, Intertemporal CAPM, GARCH, Relative downside volatility, Risk-return trade-off, Upside uncertainty.
JEL Classification: C22, C51, G12, G15

1. Introduction

The idea of a systematic trade-off between risk and returns is fundamental to the modern finance theory. Merton (1973) intertemporal capital asset pricing (ICAPM) theory asserts that there exists a positive and linear relation between the conditional variance and expected excess market returns. Yet, as Rossi and Timmermann (2009) point out, after more than two decades of research, there is little agreement regarding the basic properties of this relationship. Both Ghysels, Santa-Clara, and Valkanov (2005) and Rossi and Timmermann (2009) provide comprehensive reviews of this literature.

Recent contribution to this line of research include Ghysels, Santa-Clara, and Valkanov (2005) and Ludvigson and Ng (2007), who find a positive and significant relationship in the US data, and Brandt and Kang (2004) who find a significantly

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negative conditional relationship. Bollerslev and Zhou (2006) find an unambiguously positive relationship between returns and implied volatility, but they find that the sign of the relationship between contemporaneous returns and realized volatility depends on the underlying model parameters.

Two important underlying assumptions in the empirical risk–return trade-off literature are: (i) a constant market price of risk, and (ii) a symmetric conditional distribution for returns. Time-varying market price of risk is widely accepted in the term structure of interest rate literature; see Dai and Singleton (2002) and Duffee (2002). Moreover, asymmetry in equity market returns and volatility has a long history in the financial literature.[1] Christoffersen, Heston, and Jacobs (2006) document the presence of time-varying conditional skewness in financial time series. Jondeau and Rockinger (2003) document the existence of negative skewness, both in international equity market and in foreign exchange market returns. Harvey and Siddique (2000) show that conditional skewness captures asymmetry in the risk. Negative time-varying conditional skewness implies that extreme negative market realizations are more frequent than positive realizations.

We relax constant market price of risk and symmetric returns assumptions and derive the market price of risk analytically. By relaxing these assumptions, we find that the market price of risk is time varying and nonlinear in effective risk aversion parameters. However, in order to make our results comparable to the existing literature, we linearize the risk–return relationship and estimate the parameters of the linearized reduced form model. We define "downside risk" as the risk borne by the investor if the realized market return falls below a certain threshold. If the market return rises above the same threshold, we call it "upside uncertainty". In addition, we define the difference between downside risk and upside uncertainty as the "relative downside risk" for each time period. We find a robust positive trade-off between market relative downside risk and the conditional mode. The market price of risk needs to be positive to support a positive risk–return trade-off in market returns. In our study, the market price of risk, which is the slope coefficient in the regression of excess returns on conditional volatility, is a nonlinear function of conditional skewness.[2] The shape of this nonlinear function depends on effective risk aversion parameters. Moreover, we find that, for S&P 500 returns, the annualized average value of the estimated market price of risk is close to the estimated Sharpe ratio in annual data.

We propose a new method to study risk–return trade-off in financial market returns. First, we derive a reduced-form equilibrium relationship between risk and equity returns for a representative investor with Gul (1991) disappointment

[1] See Hansen (1994), Bekaert and Wu (2000), and Brandt and Kang (2004) for a review.

[2] In this study, we estimate Pearson mode skewness, defined as the difference between the mean and the mode divided by the standard deviation, which is a more robust measure of asymmetry in comparison with conditional skewness. For a discussion, see Kim and White (2004).

aversion preferences in an endowment economy.[3] This investor is aware of market relative downside risk, and hence demands compensation for relative downside volatility. This step is conceptually similar to the method of Ang, Chen, and Xing (2006). Second, we argue that if the investor is aware of relative downside risk, then this should be reflected in equilibrium asset prices. To be consistent with this argument, we assume that in equilibrium, logarithmic returns follow the binormal distribution of Gibbons and Mylroie (1973), explicitly disentangling downside and upside market volatilities. Under these conditions, we provide a detailed analysis of the risk–return trade-off in equilibrium. Third, to empirically examine this risk–return trade-off, we introduce a new generalized autoregressive conditional heteroskedasticity (GARCH) model, which we call binormal GARCH (BiN-GARCH). We show that this model characterizes S&P 500 and many international financial market excess returns well. Finally, we show that under binormal dynamics and using the BiN-GARCH model, the relationship between conditional mode and relative downside risk is positive and significant. These findings mean that conditional skewness is a priced factor in financial markets.

This is the first paper to explicitly model upside and downside volatilities. Our empirical findings indicate that first, on average, annualized daily downside and upside volatilities over the sample period are 17.42% and 15.56% (an average relative downside volatility of almost 2%), respectively, for S&P 500 returns.[4] Additionally, the data imply that annualized daily relative downside volatility is almost 2% for Australia, Germany, and the UK. A summary of our findings is available in Panel B, Table I. Measures of upside and downside volatility are highly correlated. We find a correlation of 0.82, which suggests co-movements in the same direction. Second, our findings shed new light on the traditional "leverage effect" of Black (1976) and Christie (1982). The leverage effect states that negative return shocks today have larger impact on future volatility than positive return shocks of similar magnitude. We find that negative shocks today have a much smaller impact on asymmetry than positive shocks of similar magnitude. This "asymmetry in asymmetry" is the opposite of what we typically observe in studies on asymmetry in volatility.

Our findings are instructive in understanding the conflicting empirical results on risk–return trade-off reported in the literature since we tie these contradictory outcomes to market asymmetry and the time-varying market price of risk. Moreover,

[3] Recently, there has been renewed interest in this class of preferences. See Routledge and Zin (2010) and Bonomo *et al.* (2011).

[4] The observation that average relative downside volatility is close to 2% does not depend on the parametric model used in estimation. In Panel B of Table 1, we present parametric and nonparametric, high-frequency estimates of relative downside volatility. Both measures deliver the same magnitude for this quantity. Standard deviations of relative downside risk in Panel B, Table 1 are generally considerably larger than the average values, implying high variation. This is not surprising given the empirical support for relative downside volatility as a pricing factor.

our results suggest that the amplitude of this trade-off increases with the value of the conditional skewness. Thus, we believe that the mixed results reported in the literature stem from forced estimation of a process by a single parameter. In particular, our empirical results support the findings of Ghysels, Santa-Clara, and Valkanov (2005), Ludvigson and Ng (2007), and Rossi and Timmermann (2009).

Our work contributes to the literature on downside risks. Ang, Chen, and Xing (2006) show that the cross section of stock returns reflects a premium for downside risk and provide a methodology for estimating this downside risk premium using daily data. We conduct a time-series study of downside risks, model and estimate downside volatility through time, and examine the relation between downside risk and measures of central tendency in asset returns. Barndorff-Nielsen et al. (2010) introduce measures of downside risk, which they call "downside realized semivariance." These measures are entirely based on downward moves, measured by using high frequency data. Building on this foundation, Andersen and Bondarenko (2009) study the properties of model-free option implied volatility and refine the notion of volatility premium based on nonparametric upside and downside semivariance measures. We rely on a GARCH framework to measure and estimate downside risk by maximum likelihood, using daily data.

Our study of high frequency data makes an important contribution to the literature on conditional skewness. Since the work of Hansen (1994), many studies have tried to propose a testing procedure to evaluate the validity of competing conditional skewness specifications.[5] We believe that we provide this crucial tool since relative downside risk is driven by conditional skewness. Thus, testing for validity of relative downside volatility specifications is in fact a test for validity of conditional skewness. Our empirical findings also show that relative downside volatility is not a proxy for other known pricing factors, such as the variance premium examined by Bollerslev, Tauchen, and Zhou (2009), but an independent and complementary determinant of the equity risk premium.

The remainder of the paper is organized as follows. In Section 2, as the first step, we present a simple theoretical model for the downside risk premium in a consumption-based equilibrium setting to motivate our empirical study. A discussion of the properties of the binormally distributed returns in equilibrium follows. In Section 3, we introduce the BiN-GARCH model. In Section 4, we discuss our data. We present and discuss our empirical findings in Section 5. These results include findings for *Standard and Poor's* 500 (S&P 500) excess returns and extensive robustness checks for international data, high frequency data, and Mincer–Zarnowitz regressions for nonparametric and BiN-GARCH measures of the downside risk. Section 6 concludes.

[5] Examples include Harvey and Siddique (1999, 2000), Jondeau and Rockinger (2003), and Brooks, Burke, Heravi, and Persand (2005), among others.

2. A Simple Equilibrium Model of Downside Risk

We consider an equilibrium consumption-based setting, with a representative investor who has a rational disappointment aversion utility function (henceforth, DA) of Gul (1991), defined over the consumption flow. This utility function embeds downside risk and allows the investor to treat downside risk and upside uncertainty differently.[6]

We further assume that in equilibrium, logarithmic returns follow a binormal distribution. This allows us to explicitly disentangle measures of downside risk and upside uncertainty in the stock market, measures that otherwise would not be separable. We then derive and analyze the implied intertemporal risk–return relationship and its sensitivity to investor's preferences. Subsequent sections empirically examine this new intertemporal risk–return tradeoff, using a large sample of market index excess returns.

2.1 PREFERENCES, STOCHASTIC DISCOUNT FACTOR, AND PRICING CONDITIONS

Formally, let V_t be the recursive intertemporal utility functional:

$$V_t = (1-\delta)C_t + \delta\mathcal{R}_t(V_{t+1}), \tag{1}$$

where C_t is the current consumption, δ is the time preference discount factor, and $\mathcal{R}_t(V_{t+1})$ is the certainty equivalent of the random future utility, conditional on time t information. In DA preferences, the certainty equivalent function, $\mathcal{R}(\cdot)$ is implicitly defined by:

$$\frac{\mathcal{R}^{1-\gamma}-1}{1-\gamma} = \int_{-\infty}^{\infty}\frac{V^{1-\gamma}-1}{1-\gamma}\mathrm{d}F(V) - \left(\frac{1}{\alpha}-1\right)\int_{\infty}^{\mathcal{R}}\left(\frac{\mathcal{R}^{1-\gamma}-1}{1-\gamma}-\frac{V^{1-\gamma}-1}{1-\gamma}\right)\mathrm{d}F(V). \tag{2}$$

The parameter α is the coefficient of disappointment aversion satisfying $0 < \alpha \leq 1$, and $F(\cdot)$ is the cumulative distribution function for the continuation value of the representative agent's utility. Several particular cases are worth mentioning. When α is equal to one, $\mathcal{R}$ becomes the certainty equivalent corresponding to expected utility and V_t represents the Kreps and Porteus (1978) preferences. When $\alpha < 1$, outcomes lower than $\mathcal{R}$ receive an extra weight $(1/\alpha - 1)$, decreasing the certainty equivalent.

[6] Ang, Chen, and Xing (2006) use a similar setup to illustrate cross-sectional pricing of the downside risk in an equilibrium setting. In their model, the utility function depends on wealth and not on consumption. We construct our model using consumption-based preferences. However, since we assume perfect elasticity of substitution along deterministic consumption paths, then the stochastic discount factor in our model only depends on market returns. This is similar to the SDF in Ang, Chen, and Xing (2006).

Thus, α is interpreted as a measure of disappointment aversion, and outcomes below the certainty equivalent are considered disappointing.[7] Figure 1 displays the differences between Kreps and Porteus (1978) preferences ($\alpha = 1$) and Gul (1991) DA preferences. It is clear that the lower the level of disappointment tolerance (smaller α values), the steeper is the indifference curve in comparison with Kreps–Porteus preferences.

The stochastic discount factor implied by the preferences is given by:

$$S_{t,t+1} = \delta(\delta R_{t+1})^{-\gamma}\left(\frac{I(\delta R_{t+1} < 1) + \alpha I(\delta R_{t+1} \geq 1)}{\mathbb{E}_t[I(\delta R_{t+1} < 1)] + \alpha\mathbb{E}_t[I(\delta R_{t+1} \geq 1)]}\right), \quad (3)$$

where $I(\cdot)$ is the indicator function that takes the value 1 if the condition is met and the value 0 otherwise, and R_{t+1} is the simple gross return on an asset that yields aggregate consumption as payoff, for which the stock market portfolio index is a proxy. For the representative investor, down markets correspond to periods where the log return, $r_{t+1} = \ln R_{t+1}$, falls below the marginal rate of time preference, $-\ln\delta$.

Our theoretical model, while based on dynamic preferences, does not directly relate the intertemporal marginal rate of substitution to the relative downside risk factor. In the general case, variations in the market price–consumption ratio contribute to the stochastic discount factor. These variations are endogenous to the model and depend on the dynamics of market downside volatility. However, by assuming perfectly elastic intertemporal substitution, we eliminate their effect to ease the model solution. The recursion in Equation (1) characterizes the Epstein and Zin (1989) recursive utility when the elasticity of intertemporal substitution is infinite, meaning that the representative agent perfectly substitutes out consumption through time. Our focus in this article is on returns and volatility. Thus, this assumption, which eliminates the effect of consumption growth rate, additionally eliminates the possibility of future volatility feeding back into current consumption through precautionary savings. As a result, the dynamics of the relative downside risk do not influence the equity premium solution. This means that the model does not address the potential hedging demands arising from time-varying relative downside risk. This issue is a subject for our future research.

Consumption and portfolio choice induces a restriction on the simple gross return of the market portfolio that is given by the Euler equation:

$$\mathbb{E}_t[S_{t,t+1}R_{t+1}] = 1, \quad (4)$$

[7] Notice that the certainty equivalent, besides being decreasing in γ, is also increasing in α. Thus α is also a measure of risk aversion, but of a different type than γ.

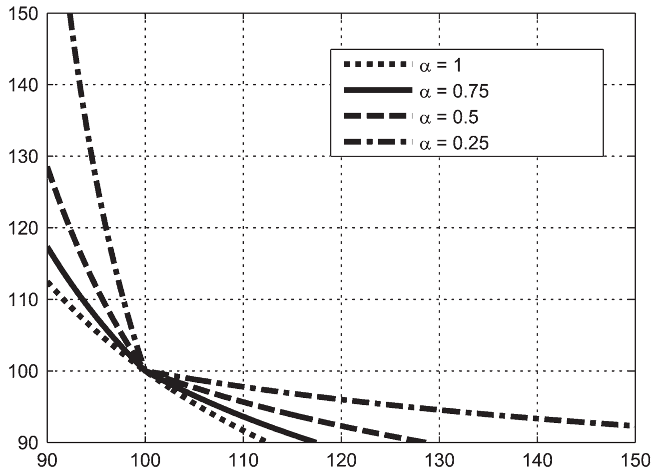

Figure 1. Disappointment aversion preferences and Kreps–Porteous Preferences
This figure depicts the indifference curves over two outcomes. $\alpha = 1$ represents the familiar Kreps and Porteus (1978) preferences. All other values of α represents Gul (1991) disappointment aversion preferences. This figure shows that DA preferences with lower tolerance for disappointment, that is, smaller values for α, have steeper slopes for indifference curves in comparison with Kreps and Porteus preferences.

where the stochastic discount factor $S_{t,t+1}$ is given by Equation (3). Once we know how the equilibrium returns are distributed, then Equation (4) constitutes the basis for studying the equilibrium risk–return tradeoff.

2.2 EQUILIBRIUM RETURN DISTRIBUTION AND THE RISK–RETURN TRADEOFF

We use the binormal distribution introduced by Gibbons and Mylroie (1973) to model logarithmic returns in equilibrium. It is an analytically tractable distribution, which accommodates empirically plausible values of skewness and kurtosis, and nests the familiar Gaussian distribution.[8] We assume that logarithmic returns, r_{t+1}, follow a binormal distribution with parameters $(m_t, \sigma_{1,t}, \sigma_{2,t})$ conditional on information up to time t. The conditional density function of r_{t+1} is given by:

$$f_t(x) = A_t \exp\left(-\frac{1}{2}\left(\frac{x - m_t}{\sigma_{1,t}}\right)^2\right) I(x < m_t) + A_t \exp\left(-\frac{1}{2}\left(\frac{x - m_t}{\sigma_{2,t}}\right)^2\right) I(x \geq m_t),$$

[8] See Bangert, Goodhew, Jeynes, and Wilson (1986), Kimber and Jeynes (1987), and Toth and Szentimrey (1990), among others, for examples of using the binormal distribution in data modeling, statistical analysis and robustness studies.

where m_t is the conditional mode and $A_t = \sqrt{2/\pi}/(\sigma_{1,t} + \sigma_{2,t})$. Figure 2 compares the probability density functions of standard normal and binormal distributions. It is immediately obvious that with smaller values of Pearson mode skewness, the left tail of the binormal distribution becomes increasingly "fatter" than standard normal, and the right tail becomes "slimmer." Also notice that smaller values for skewness, and hence Pearson mode skewness, cause a rightward movement of the probability peak, which is inline with the assumption of unequal treatment of downside and upside volatility.[9]

We notice that m_t is the conditional mode, and up to a multiplicative constant, $\sigma^2_{1,t}$ and $\sigma^2_{2,t}$ are interpreted as conditional variances of returns, conditional on returns being less than the mode (downside variance), and conditional on returns being greater than the mode (upside variance), respectively. Specifically,

$$\mathrm{Var}_t[r_{t+1}|r_{t+1} < m_t] = \left(1 - \frac{2}{\pi}\right)\sigma^2_{1,t} \quad \text{and} \quad \mathrm{Var}[r_{t+1}|r_{t+1} \geq m_t] = \left(1 - \frac{2}{\pi}\right)\sigma^2_{2,t}. \tag{5}$$

We consider this property to be the most important characteristic of the binormal distribution, given our objectives in this project.

Binormal distribution can be parameterized by the mean μ_t, the variance σ_t^2, and the Pearson mode skewness p_t. Binormal distribution implies the following functional forms for these statistics:

$$\begin{aligned} \sigma_t^2 &= (1 - 2/\pi)\left(\sigma_{2,t} - \sigma_{1,t}\right)^2 + \sigma_{1,t}\sigma_{2,t} \\ p_t &= \sqrt{2/\pi}(\sigma_{2,t} - \sigma_{1,t})/\sigma_t \\ \mu_t &\equiv m_t + \sigma_t p_t = m_t + \sqrt{2/\pi}(\sigma_{2,t} - \sigma_{1,t}). \end{aligned} \tag{6}$$

It can be shown that the initial parameters $\sigma_{1,t}$ and $\sigma_{2,t}$ are expressed in terms of the total variance and the Pearson mode skewness as follows:

$$\begin{aligned} \sigma_{1,t} &= \sigma_t\left(-\sqrt{\pi/8}p_t + \sqrt{1 - (3\pi/8 - 1)p_t^2}\right) \\ \sigma_{2,t} &= \sigma_t\left(\sqrt{\pi/8}p_t + \sqrt{1 - (3\pi/8 - 1)p_t^2}\right), \end{aligned} \tag{7}$$

which implies a bound on the Pearson mode skewness: $|p_t| \leq 1/\sqrt{\pi/2 - 1} \approx 1.3236$.

Assuming that log returns are conditionally binormally distributed, we still need the conditional moment generating function $M_t(u) = \mathbb{E}_t[\exp(ur_{t+1})]$ as

[9] We want a simple parametric framework that directly allows us to model downside and upside volatility. There are many candidates, but binormal distribution has the advantage of simplicity and tractability. In this regard, we follow the example of Bollerslev, Tauchen, and Zhou (2009), where they assume conditional normality for stock returns to highlight the role of the variance risk premium.

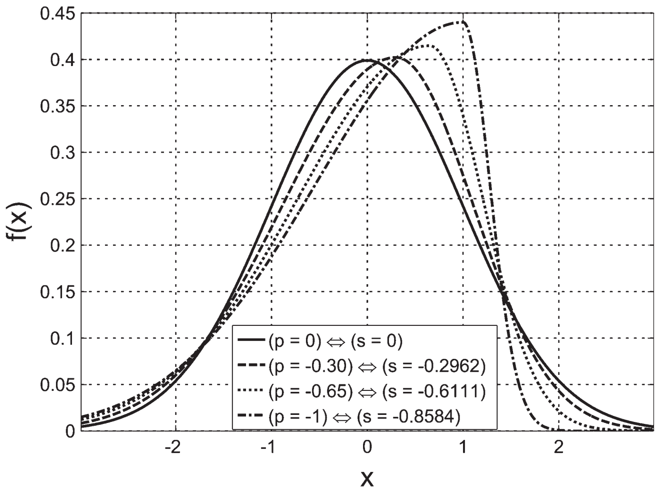

Figure 2. Standard normal and binormal marginal density functions
This figure compares the standard normal and binormal marginal density functions ($f(X)$) for different values of Pearson mode skewness and the corresponding skewness values. The plot with $p = s = 0$ and drawn in solid line is the standard normal density function.

well as the conditional truncated moment generating function $M_t(u;x) = \mathbb{E}_t[\exp(ur_{t+1})I(r_{t+1} \geq x)]$ of returns to be able to explicitly characterize the Euler equilibrium restriction (4). To save space and avoid a lengthy exposition, these functions are explicitly given in the external appendix. However, we show that the Euler Equation (4) can also be represented by a nonlinear restriction, say

$$G(m_t, \sigma_{1,t}, \sigma_{2,t}) = 0, \tag{8}$$

on the parameters $(m_t, \sigma_{1,t}, \sigma_{2,t})$ of the conditional distribution of log returns. The mode is then derived as a function of downside and upside volatilities from Equation (8).

The nonlinear function G is explicitly known and given by:

$$\begin{aligned} G(m_t, \sigma_{1,t}, \sigma_{2,t}) &= \delta^{1-\gamma}\frac{M_t(1-\gamma) + (\alpha-1)M_t(1-\gamma;-\ln\delta)}{1+(\alpha-1)M_t(0;-\ln\delta)} - \\ &= \frac{\tilde{M}_t(1-\gamma) + (\alpha-1)\tilde{M}_t(1-\gamma;0)}{1+(\alpha-1)\tilde{M}_t(0;0)} - 1, \end{aligned} \tag{9}$$

where $\tilde{M}_t(u)$ and $\tilde{M}_t(u;x)$ are, respectively, the moment generating function and a the truncated moment generating function of $r_{t+1} + \ln\delta$. Since $r_{t+1} + \ln\delta$ has a binormal distribution with parameters $(m_t + \ln\delta, \sigma_{1,t}, \sigma_{2,t})$, we can conclude that m_t and δ are not separately identifiable from Equation (9). The restriction (8) implies that the conditional mode is in fact an implicit nonlinear function of conditional downside and upside volatilities, $m_t = -\ln\delta + g(\sigma_{1,t}, \sigma_{2,t})$. The implicit function g is parameterized by the preference parameters γ and α only.

The equation $m_t = -\ln\delta + g(\sigma_{1,t}, \sigma_{2,t})$ defines a new risk–return relation that relates the conditional mode to the conditional downside and upside volatilities. To be able to deal with this new trade-off between risk and reward, we first-order linearize the nonlinear restriction (8) around the steady-state values $(\bar{\sigma}_1, \bar{\sigma}_2)$ to obtain

$$m_t = -\ln\delta + g(\sigma_{1,t}, \sigma_{2,t}) \approx \lambda_0 + \lambda_1\sigma_{1,t} + \lambda_2\sigma_{2,t}, \tag{10}$$

where

$$\begin{aligned}\lambda_1 &= -\frac{G_{\sigma_1}(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)},\ \lambda_2 = -\frac{G_{\sigma_2}(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1, \bar{\sigma}_2), \bar{\sigma}_1, \bar{\sigma}_2)},\\ \text{and } \lambda_0 &= \bar{m} - \lambda_1\bar{\sigma}_1 - \lambda_2\bar{\sigma}_2,\end{aligned} \tag{11}$$

and where $\bar{m} = -\ln\delta + g(\bar{\sigma}_1, \bar{\sigma}_2)$ and, $G_m(\cdot,\cdot,\cdot)$, $G_{\sigma_1}(\cdot,\cdot,\cdot)$ and $G_{\sigma_2}(\cdot,\cdot,\cdot)$ denote the first-order partial derivatives of the function $G(\cdot,\cdot,\cdot)$ with respect to its arguments, m, σ_1 and σ_2, respectively. These results are based on applying the well-known implicit function theorem to Equation (8).

Equation (10) clearly shows that λ_1 and λ_2 depend on γ and α, but not on δ. In equilibrium, only effective risk aversion (i.e. the (γ, α) pair in this case) determines the sensitivity of reward to risk, which is typical in the risk–return trade-off literature. Figure 3 presents the relationship between λ_1 and λ_2 and structural parameters, α and γ. We compute the values of upside and downside volatility premia, given analytical solutions for the first two expressions in Equation (11) and theoretically plausible values for α and γ.[10] The figure clearly shows that the lower the disappointment aversion (high α), the smaller the values of both λ_1 and λ_2. Similarly, the size of risk aversion parameter, γ, increases the sensitivity of the conditional mode to upside and downside volatilities, for any given value of disappointment aversion.

It is well known that mode is more robust to outliers than the mean. This property holds for conditional mode and expected values. Mode is also an interesting measure of reward as it represents the most likely realization of returns. Thus, disappointment averse investors view the conditional mode to be at least as informative as the expected value of returns. Given the expression (10), the traditional risk–return trade-off that relates expected returns to the total variance may be expressed as:

$$\mu_t = m_t + \sigma_t p_t = \lambda_0 + \lambda_t^* \sigma_t, \tag{12}$$

[10] Full derivation of relationships in Equation (11) for λ_1 and λ_2 in terms of structural parameters, γ and α, is long, tedious, and is done using Maple.

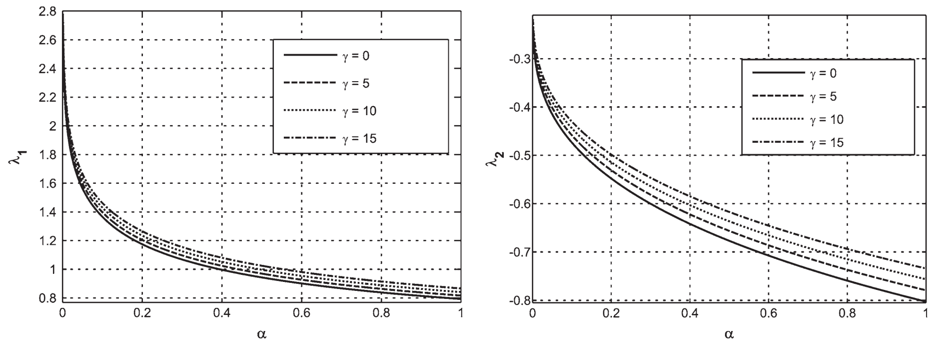

Figure 3. The relationship between downside and upside volatility premia and structural parameters
This figure depicts the relationship between downside volatility premium, λ_1, upside volatility premium, λ_2, and structural parameters α and γ, which represent disappointment aversion and risk aversion coefficients, respectively. The values plotted in this figure are based on the analytical solutions to $\lambda_1 = -\frac{G_{\sigma_1}(g(\bar{\sigma}_1,\bar{\sigma}_2),\bar{\sigma}_1,\bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1,\bar{\sigma}_2),\bar{\sigma}_1,\bar{\sigma}_2)}$ and $\lambda_2 = -\frac{G_{\sigma_2}(g(\bar{\sigma}_1,\bar{\sigma}_2),\bar{\sigma}_1,\bar{\sigma}_2)}{G_m(g(\bar{\sigma}_1,\bar{\sigma}_2),\bar{\sigma}_1,\bar{\sigma}_2)}$ and computed for plausible values of α and γ.

where

$$\lambda_t^* = (1 - (\lambda_1 - \lambda_2)\sqrt{\pi/8})p_t + (\lambda_1 + \lambda_2)\sqrt{1 - (3\pi/8 - 1)p_t^2}. \qquad (13)$$

The first equality in Equation (12) follows by the definition of mean in binormal distribution, Equation (6). The second equality in Equations (12) and (13) follow from Equations (7) and (10). Equation (12) characterizes the traditional risk–return trade-off in this model and shows that the price of risk depends on the asymmetry in returns.

The impact of skewness on the traditional risk–return relation is clearly visible in Figure 4. The traditional linear risk–return tradeoff of Merton (1973) corresponds to skewness equal to zero, which is depicted by the solid line in Figure 4. In this figure, we set disappointment aversion coefficient, α, to be equal to 1. This means that there is no disappointment aversion in the model and the investor has Kreps–Porteus preferences. Kreps–Porteus preferences are symmetric. Hence, only asymmetry in returns matters for the risk–return tradeoff. If the risk aversion coefficient (γ) is less than 1, as in the left-hand side panel, then the relation between equity premium and conditional volatility strengthens with increasing positive skewness, and weakens with increasing negative skewness. The middle panel shows that when risk aversion coefficient equals 1, then skewness has no impact on the the risk–return relationship. The right-hand side panel shows that when the coefficient of risk aversion is greater than 1, then the equity premium–volatility relation weakens with increasing positive skewness and strengthens with increasing negative skewness. In our opinion, since skewness is time varying, Figure 4 sheds some light on the empirically inconclusive results seen in the literature.

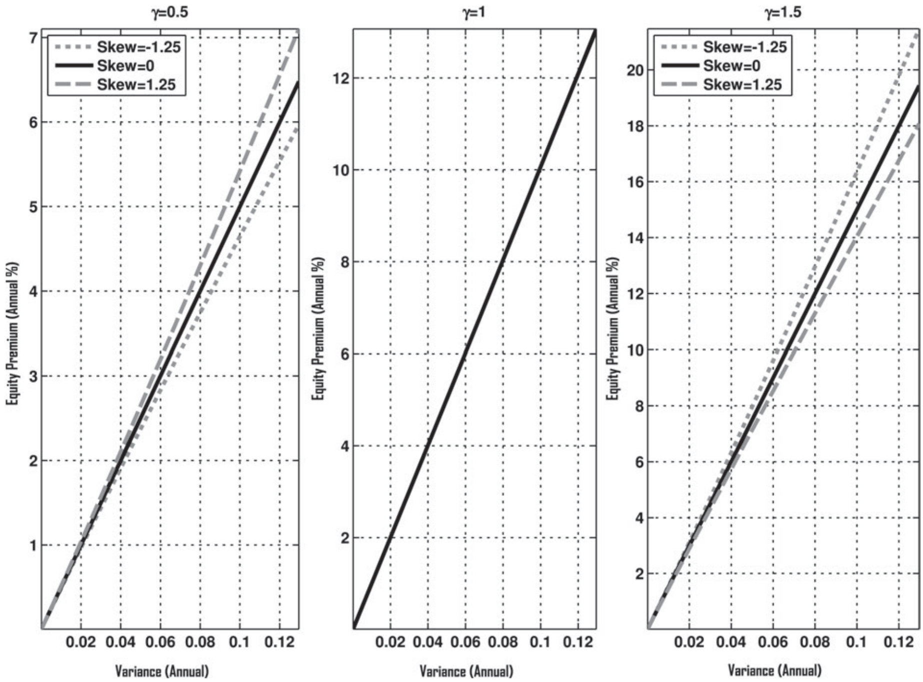

Figure 4. The impact of skewness on risk–return tradeoff when $\alpha = 1$
This figure depicts the impact of skewness on the traditional risk–return relationship of Merton (1973). The values on the vertical axis represent annualized equity premium in percentage. Values on the horizontal axis represent annualized volatility. The case studied by Merton corresponds to normally distributed returns (skewness $= 0$, solid line in the figure). Asymmetries in this figure are solely due to asymmetry in distribution, since by setting $\alpha = 1$, we shut down the channel for disappointment aversion in preferences. With $\gamma < 1$, negative values of skewness, which imply negative asymmetry in returns, weaken the risk–return relation. On the other hand, positive values of skewness strengthen this relationship. With $\gamma = 1$, skewness does not have an impact on the risk–return relationship. Negative values of skewness, which imply negative asymmetry, strengthen the risk–return relation, when $\gamma > 1$. Positive values of skewness, on the other hand, weaken this relationship.

3. Conditional Mode and Pearson Mode Skewness

3.1 BiN-GARCH MODEL SPECIFICATION

We allow for time variation in the return distribution. Specifically, we allow for heteroskedasticity dynamics similar to GARCH models, besides we directly model the mode and the Pearson mode skewness of the conditional return distribution. This is where our work differs from existing competing models. We rely on conditional mode and Pearson mode skewness to model central tendency and asymmetry since they are less sensitive to outliers than mean and skewness.[11]

[11] See Kim and White (2004) for a detailed discussion.

We assume that, conditional on information up to time t, returns r_{t+1} follow a binormal distribution with mode m_t, variance σ_t^2 and Pearson mode skewness p_t. In line with the literature, we allow for the negative correlation between volatility and returns or the so-called "leverage effect," where firms' leverage increases with negative returns. We borrow our specification for heteroskedasticity from the NGARCH model of Engle and Ng (1993),

$$\sigma_{t+1}^2 = \beta_0 + \beta_1\sigma_t^2 + \beta_2\sigma_t^2(z_{t+1} - \theta)^2, \tag{14}$$

where $z_{t+1} = (r_{t+1} - \mathbb{E}_t[r_{t+1}])/\sigma_t$ are standardized residuals. As a result, our specification nests NGARCH.[12] Christoffersen and Jacobs (2004) show that NGARCH has a better out-of-sample performance in option pricing compared to several alternative GARCH models.

Given that the Pearson mode skewness is bounded $\left(|p_t| \leq 1/\sqrt{\pi/2 - 1}\right)$, we use the hyperbolic tangent transformation to (i) guarantee the bounds, and (ii) to preserve the direction of variation such that we can directly interpret the estimated parameters, following Hansen (1994) and Jondeau and Rockinger (2003). As a result, we assume that the Pearson mode skewness evolves following:

$$p_{t+1} = \sqrt{\frac{2}{\pi - 2}}\tanh\left(\kappa_0 + \kappa_1 z_{t+1}^* I\left(z_{t+1}^* \geq 0\right) + \kappa_2 z_{t+1}^* I\left(z_{t+1}^* < 0\right) + \kappa_3 p_t\right), \tag{15}$$

where $z_{t+1}^* = (r_{t+1} - m_t)/\sigma_t$. Accordingly, κ_1, κ_2, and κ_3, measure the impact of positive and negative shocks, as well as persistence on p_t. Our formulation for skewness is an extension of the model developed in Harvey and Siddique (1999). This nonlinear GARCH-type dynamics of the conditional Pearson mode skewness also features asymmetry in asymmetry. Asymmetries in the Pearson mode skewness are generated by deviations of realized returns from the conditional mode. We recall that dynamics of volatility and Pearson mode skewness lead to direct downside and upside volatility modeling through Equation (7).

Alternatively, it is possible to specify the dynamics of $\sigma_{1,t}$ and $\sigma_{2,t}$. However, we want to rely on well-known dynamics for volatility (σ_t). Many authors use GARCH dynamics for conditional variance and this choice characterizes the returns well. In our model, upside uncertainty and downside risk, together characterize the (total) volatility. Hence, we want them to give rise to NGARCH dynamics for σ_t.

Following the linear approximation in Equation (10), we specify the conditional mode as:

[12] Our empirical findings do not rely on NGARCH-type dynamics.Assuming EGARCH-type dynamics of Nelson (1991), we find very similar results.

$$m_t = \lambda_0 + \lambda_1 \sigma_{1,t} + \lambda_2 \sigma_{2,t}. \tag{16}$$

This specification of the conditional mode, GARCH-in-Mode, is motivated by the equilibrium model of Section 2, and is analogous to the ARCH-in-Mean model of Engle, Lilien, and Robins (1987), which relates expected returns to volatility. We recall from Section 2.2 that $\sigma_{1,t}$ and $\sigma_{2,t}$ are defined by Equations (5) and (7).

The mode, similar to the mean, also characterizes the central tendency. Hence, we assume that in Equation (16), the future conditional mode has a linear relationship with upside or downside volatilities of returns, depending on whether return realizations are above or below the current conditional mode.

3.2 BiN-GARCH AND RISK–RETURN TRADE-OFF

Based on the ICAPM model of Merton (1973), the vast majority of studies focus on verifying a positive (linear) relationship between the conditional expected excess return of the stock market and the market's conditional variance through estimation of a time-invariant market price of risk.

In what follows, we propose an alternative to the conditional mean and conditional variance relationship as a measure for risk–return trade-off in empirical tests. As discussed above, for negatively asymmetric returns with outliers, and assuming a time-varying market price of risk, we build our testing procedure for risk–return trade-off based on a relationship between the conditional mode and the conditional downside and upside variances. The basis of our proposal is the relationship between the conditional mode and the conditional mean in Equation (16).

First, if both the conditional mode and the conditional Pearson mode skewness are constant, the first equality in Equation (12) implies that they are respectively the drift and the slope of the linear regression of returns onto the conditional volatility. In this case, a negative Pearson mode skewness implies that expected returns fall in response to an increase in volatility. Consequently, the positive linear relationship between expected returns and volatility, as suggested by Merton's (1973) ICAPM, would be inconsistent with the fact that both the conditional mode and the conditional Pearson mode skewness are constant and the latter is negative.

Second, based on Ang, Chen, and Xing (2006), it is clear from Equation (5) that $\sigma_{1,t}$ and $\sigma_{2,t}$ are, respectively, the measures of market downside and upside volatilities using the conditional mode of returns as the cutoff point. If equity is more volatile in a bear market than it is in a bull market, then investors require a compensation for holding it, since equity tends to have low payoffs when they feel poor and pessimist, compared to when they feel wealthy and confident.[13] This is in line

[13] Explicitly, this statement translates into $\sigma_{1,t} > \sigma_{2,t}$.

with what Cochrane (2007) points out about the relationship between equity premium and business cycles.

So far, we have shown that our theoretical results imply a positive relationship between the conditional mode and the relative downside risk. This result does not contradict the conventional risk–return equation used in the literature. As discussed in Section 2.2, we can rewrite the expected return as:

$$\mathbb{E}_t[r_{t+1}] = \lambda_0 + \lambda_t^* \sigma_t, \tag{17}$$

where λ_t^* is defined in Equation (13). This implies a time-varying price of risk, which depends on the conditional asymmetry. This relationship is similar to the typical equation seen in the literature, for example, in Ghysels, Santa-Clara, and Valkanov (2005), except for time variation in the market price of risk.

4. Data

We use S&P 500 index excess returns and *Morgan Stanley Capital International* (MSCI) daily market index excess returns for five major markets obtained from *Thomson Reuters Datastream*. We use USD-denominated MSCI indices in order to have comparable results. All these series start in January 1980 and end in December 31 2009.

Table I reports summary statistics of the data. Annualized return means and standard deviations in percentages are reported in the fourth and the fifth columns. We report unconditional skewness in column six. We observe negative unconditional skewness for all the market returns. The value of unconditional skewness is not small relative to the average daily returns. All series seem to be highly leptokurtotic since they all have significant unconditional excess kurtosis. The reported p values of Jarque and Bera (1980) normality test imply significant departure from normality in all series. Our proxy for the risk-free rate is the yield of the 3-month constant maturity US Treasury Bill, which we obtained from the Federal Reserve Bank of St Louis FRED II data bank. The crash of October 1987, the Asian crisis of 1997, and the Russian default of 1998 episodes are represented in the data. All data series include the 2007–09 Great Recession.

Our intraday data series come from Olsen Financial Technologies and are their longest available 1-minute close level S&P 500 index price series. This data set spans the period from February 1986 to September 2010. To reduce the market microstructure effect in our empirical results, we construct intradaily returns at frequencies lower than 1 minute. VIX data are freely available from Chicago Board of Trade's website. Data on monthly variance risk premium are from Hao Zhou's website.

Table I. Summary statistics of the data

The top panel of this table reports summary statistics of excess returns. Calculation of the returns is based on subtracting daily 3-months U.S. Treasury Bill rate from the log difference of the market total return index in each market. Mean of excess returns and standard deviations are reported as annualized percentages. Excess kurtosis values are reported. The column titled ''J-B p-Value'' reports p values of Jarque and Bera (1980) test of normality in percentages. The bottom panel reports the computed statistics of the observed relative downside volatility (RDV) in the data. All the results are based on fitting the full BiN-GARCH model to the data, except for S&P 500 (NP). Reported RDV is based on the mean difference between filtered downside and upside volatilities. The first column reports the annualized RDV, and the second column is the standard deviation of this quantity. Due to availability of high frequency data for S&P 500 returns, we also report the nonparametric estimate for relative downside volatility for the USA denoted as S&P 500 (NP), based on 15-minute returns. The sample period is January 1 1980 to December 31 2009. Source: Thomson Reuters Datastream and FRED II data bank at the Federal Reserve Bank of St Louis.

Panel A: Descriptive statistics, excess returns

Return series	Mean (%)	SD (%)	Skewness	Kurtosis	J-B p value
S&P 500	3.48	21.94	−1.24	31.87	0.01
Australia	0.69	27.98	−3.36	71.32	0.01
Germany	2.31	27.41	−0.23	9.49	0.01
Japan	0.36	27.11	−0.09	11.07	0.01
Switzerland	4.47	22.43	−0.24	9.27	0.01
UK	1.88	24.24	−0.32	12.33	0.01

Panel B: Descriptive statistics, relative downside volatility

	RDV (%)	SD (%)	Skewness	Kurtosis	J-B p Value
S&P 500	1.88	3.35	6.68	148.22	0.01
S&P 500 (NP)	1.84	8.71	1.21	16.36	0.01
Australia	2.25	5.16	14.41	451.99	0.01
Germany	2.18	2.89	4.28	50.29	0.01
Japan	0.19	2.42	7.50	218.74	0.01
Switzerland	1.40	2.42	5.67	77.69	0.01
UK	2.19	2.22	3.62	44.67	0.01

Figure 5 plots the time series for the relative downside volatility in S&P 500 returns in 1980–09 period, along with the upper and lower 95% confidence bands.

5. Empirical Results

5.1 BiN-GARCH FITTING OF S&P 500 EXCESS RETURNS

We now turn our attention to maximum likelihood estimation of the BiN-GARCH model, introduced in Section 3, and discuss the results. Our first step is to study the ability of different BiN-GARCH specifications in capturing the dynamics of the financial time series. We then perform extensive robustness testing. Thus, we first fit the S&P

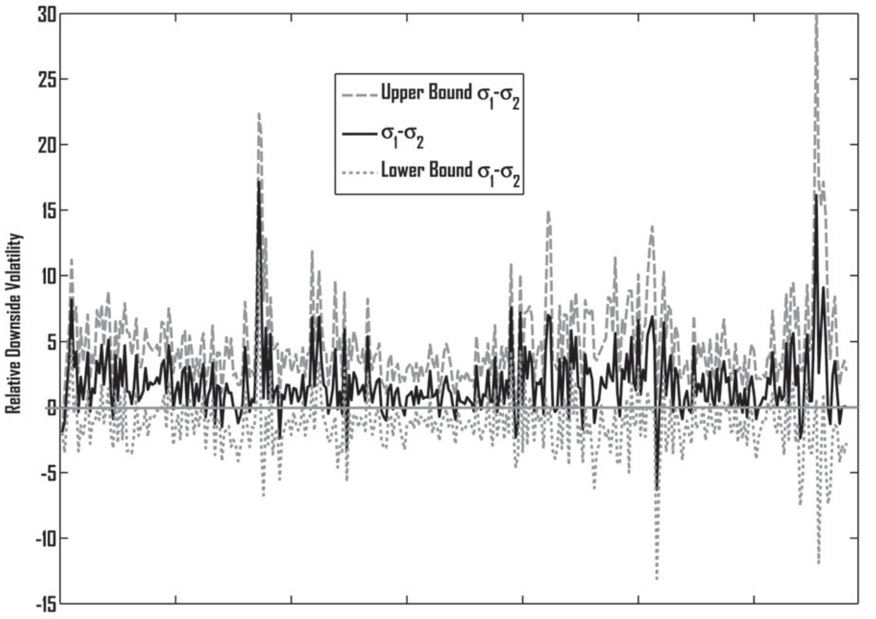

Figure 5. Relative downside volatility for S&P 500 excess returns, 1980–09
The solid line in this figure shows the annualized daily relative downside volatility, the difference between downside risk and upside uncertainty, for S&P 500 excess returns over 3-month treasury bill yields. The computed values are based on fitting the full BiN-GARCH model to daily returns starting in January 1980 and ending on December 2009. The black solid line represents computed relative downside volatility values. We plot every 22nd computed value. We use dashed and dotted lines to represent the upper and lower 95% confidence bands around relative downside volatility values. Source: Thomson Reuters Datastream and FRED II data bank at the Federal Reserve Bank of St Louis.

500 excess returns using five BiN-GARCH specifications.[14] We then use the best model to conduct the risk–return trade-off study. Our metrics for the best fit are the likelihood ratio (LR) tests against the benchmark model and the other specifications studied.

In this study, the canonical NGARCH model of Engle and Ng (1993) is the benchmark for model comparison. Estimated parameters of NGARCH model are reported under Specification (I) in column 2 of Table II. With NGARCH specification for returns, the conditional Pearson mode skewness is zero and the mode, which in this case is equal to the mean, is constant.

As is seen in Table II, LR tests indicate that all other models studied are preferred to NGARCH. Similarly, LR tests of Models II to V against each preceding model indicates that the richer models are preferred to the simpler models introduced. The

[14] By setting Pearson mode skewness equal to zero, BiN-GARCH nests Engle and Ng (1993).

Table II. ML estimation of various BiN-GARCH models using S&P 500 daily excess returns

This table presents maximum likelihood estimation results for different specifications of the binormal GARCH model for S&P 500 daily excess returns. The sample spans continuously compounded value-weighted returns on the S&P 500 Index from January 2 1980 to December 31 2009. Standard errors are given in parentheses. *, †, and ‡ denote statistical significance at 1%, 5%, and 10% levels, respectively. LR test statistics labeled as "LR Test Stat (1)" are computed with respect to the benchmark NGARCH model of Engle and Ng (1993), represented as Model (I). Likelihood ratio test statistics labeled as "LR Test Stat (2)" are computed with respect to the preceding model. Model (II) relaxes the symmetry assumption by allowing a nonzero p. Model (III) allows for time-varying p. Model (IV) imposes $\lambda_2 = -\lambda_1$ restriction. Model (V) allows for time-varying conditional mode and relaxes the $\lambda_2 = -\lambda_1$ restriction. Model (V) is the full BiN-GARCH model discussed in Section 3.

Estimated parameter	(I)	(II)	(III)	(IV)	(V)
m	4.65E–05* (1.65E–05)	0.0009* (0.0002)	0.0008* (0.0002)		
λ_0				0.0002‡ (0.0001)	−0.0004 (0.0003)
λ_1				0.6756* (0.0642)	0.6608* (0.0585)
λ_2					−0.5805* (0.0751)
β_0	1.56E–06* (2.11E–07)	1.15E–06* (2.02E–07)	1.18E–06* (2.02E–07)	1.35E–06* (1.96E–07)	1.62E–06* (2.60E–07)
β_1	0.8841* (0.0086)	0.8920* (0.0081)	0.8835* (0.0092)	0.8855* (0.0086)	0.8796* (0.0097)
β_2	0.0642* (0.0051)	0.0628* (0.0050)	0.0607* (0.0050)	0.0623* (0.0050)	0.0621* (0.0050)
θ	0.7955* (0.0661)	0.8078* (0.0668)	0.9167* (0.0828)	0.8370* (0.0749)	0.8546* (0.0777)
p		−0.1138* (0.0187)			
κ_0			−0.0628* (0.0158)	−0.0752* (0.0209)	−0.0815* (0.0197)
κ_1			0.0710* (0.0167)	0.1449* (0.0347)	0.1568* (0.0318)
κ_2			0.0187‡ (0.0105)	0.0725* (0.0172)	0.0680* (0.0167)
κ_3			0.2603† (0.1043)	0.2903* (0.0758)	0.2842* (0.0737)
Diagnostic Measures					
Log-Lik	24,639.54	24,657.87	24,673.16	24,699.76	24,701.69
LR test Stat (1)		36.67*	67.24*	120.46*	124.31*
LR test Stat (2)		36.67*	30.58*	53.20*	3.86

only exception are Models IV and V. LR test statistic indicates that these two models are indistinguishable from each other.[15]

We depart from the NGARCH model by allowing a constant, but nonzero, Pearson mode skewness in Specification (II). Parameter estimates of the model are reported in Column 3 of Table II. The estimated value of the constant conditional Pearson mode skewness is -0.1138. Results for this specification confirms that S&P 500 index returns are conditionally negatively skewed. The gain in likelihood resulting from the inclusion of a single parameter from (II) to (I), the associated LR test statistic of 36.67 and the information criterion all indicate that the NGARCH with *i.i.d.* Gaussian standardized residuals is rejected in favor of the GARCH with constant skewness at the 1% significance level or better. Estimates of constant conditional mode and Pearson mode skewness are, respectively, positive and negative and strongly significant. As discussed in Section 3, this leads to a negative relationship between expected returns and volatility. A positive risk–return relation would simply mean that either the mode or the Pearson mode skewness is misspecified, or both. This is an important result that underpins our study of risk–return trade-off based on GARCH-in-Mode estimations.

In Specification (III), we keep the mode constant and allow the Pearson mode skewness to vary over time and follow the nonlinear autoregressive dynamics specified in Equation (16). We report the estimated parameters of the specification (III) in the fourth column of Table II. All parameters are strongly significant and the inclusion of three more parameters compared to Specification (II) induces a substantial gain in likelihood. The corresponding LR test statistic of 67.24 and information criterion also strongly reject the NGARCH model in favor of Specification (III). Moreover, based on the difference in log-likelihoods between Specifications (II) and (III), we find that LR test statistic of 30.5696, which is statistically significant at the 5% level or better, leads us to favor the autoregressive conditional asymmetry specification over constant Pearson mode skewness.

These results further suggest that realizations of returns relative to the conditional mode have different impacts on conditional asymmetry as measured through the Pearson mode skewness. Estimates of κ_1 and κ_2 are both positive and κ_1 is three times more than κ_2. Thus, increases in the Pearson mode skewness due to realization of returns above the conditional mode are significantly larger than the reductions in the Pearson mode skewness due to realization of equal absolute value-sized returns below the conditional mode.

[15] Hansen (1994) uses a skewed Student's *t*-test density for modeling conditional skewness and kurtosis in his autoregressive conditional density method. We also fit the S&P 500 and international returns using an NGARCH model with skewed Student's-*t* errors. Since BiN-GARCH does not nest this model, we cannot rely on likelihood ratio tests to compare them. However, BIC values imply that BiN-GARCH is at least as good as NGARCH with skewed-*t* errors in fitting the data.

Equation (17) links the conditional mode to upside and downside volatilities. With $\lambda_2 \approx -\lambda_1$, then the mode is a function of the relative downside volatility, $\sigma_{1,t} - \sigma_{2,t}$, that is,

$$m_t \approx \lambda_0 + \lambda_1(\sigma_{1,t} - \sigma_{2,t}). \tag{18}$$

We test the Specification (IV) since we find the model-implied absolute values of λ_1 and λ_2 to be close to each other and they have the opposite sign for the lower but significant level of disappointment aversion (Figure 3).

In Specification (IV), we relax the fixed mode assumption maintained in Specifications (I–III). Estimation results for Specification (IV), which means fitting Equation (18) to the data, are reported in column 5 of Table II. In comparison with Specification (III), there is only one meaningful restriction imposed on Specification (IV): $\lambda_2 = -\lambda_1$. However, this linear restriction seems reasonably valid. This is due to the observation that first, LR test statistic of 120.45 implies that Specification (IV) is statistically preferable to the baseline NGARCH model at the 1% significance level or better. Second, in comparison with Specification (III), Specification (IV) is preferred since this model induces gains in likelihood that are not due to the inclusion of additional parameters. This is attested by LR test statistic of 53.21, which is statistically significant at the 5% confidence level or better. Estimated parameters are all statistically significant at conventional confidence levels.

The estimated parameters of the full BiN-GARCH model, Specification (V), are reported in column 6 of Table II. Again, all estimated parameters are significant at conventional levels except for λ_0, the drift in the conditional mode. As is seen in the table, this specification is readily preferable to the baseline NGARCH model based on the LR test. In comparison with Specification (IV), first notice that while we have relaxed the $\lambda_2 = -\lambda_1$ restriction, the values of estimated λ_1 and λ_2 are reasonably close and have opposite signs, thus confirming the validity of the negative impact in the time-varying conditional mode, explored in Specification (IV).[16] Second, in comparison with Specification (III), responses of the asymmetry to return realizations above and below the conditional mode increase when the conditional mode becomes time varying. Notice that the estimates of κ_1 and κ_2 are more than twice their respective values when the mode is time invariant. The LR test at the 1% level rejects Specifications (I) to (III) in the table against Specification (IV) and against the full BiN-GARCH specification. This test fails to reject Specification (IV) against Specification (V), implying that the two models are statistically equivalent.

In what follows, we use Specification (V) as our estimation model of choice. The reason for this selection is two-fold. First, we estimate both structural parameters of

[16] They have the same order of magnitude, and the ratio of their absolute values is close to 0.88. Crucially, the absolute values of the estimated parameters are quite close, less than one standard error apart. Thus, one can reasonably infer that specifications (IV) and (V) are statistically equivalent.

the preference function, α and γ, as well as the premia for downside risk and upside uncertainty, λ_1 and λ_2. In order to identify both α and γ, we need both relationships in Equation (11). This requirement forces us to estimate the full model. Second, we propose relative downside premium as another factor to be priced. Assuming only asymmetry in returns and using standard Kreps and Porteus (1978) preferences would not allow us to separate upside and downside volatility premia (in other words, relative downside risk premium). Bollerslev, Tauchen, and Zhou (2009) use Epstein–Zin–Weil preferences to separate the volatility premium from the market risk premium. The reason is that Epstein and Zin (1989) preferences have one more preference parameter, ψ (intertemporal elasticity of substitution), in comparison with the standard Kreps and Porteus (1978) preferences. Similarly, we need the disappointment aversion parameter, α, in addition to the coefficient of risk aversion, γ, to separate relative downside risk premia in our empirical procedures.

In Figure 6, we provide graphical representations for the contribution of relative downside risk for S&P 500 index daily excess returns. The annualized daily volatility and the daily Pearson mode skewness are plotted in panels A and B of Figure 6. On average, the annualized daily volatility for the sample period is 16.06%. The daily Pearson mode skewness is -0.1289. This value closely matches the estimated parameter for the *i.i.d.* Specification (II) in column 3 of Table II. Fluctuations in the Pearson mode skewness show that, although the conditional asymmetry is centered to a negative value, stock returns can be positively skewed. This contrasts with the IG-GARCH model of Christoffersen, Heston, and Jacobs (2006), which imposes a negative conditional skewness over time. Instead, the direction of asymmetry in the BiN-GARCH model is determined by relative downside volatility. Returns are negatively skewed only if equity is more volatile in a declining market than in a rising market, and are positively skewed otherwise. Finally, we present the filtered downside and upside volatility series in panels C and D of Figure 6. On average, annualized daily downside and upside volatilities over the sample period are 17.42% and 15.56% (an average relative downside volatility of almost 2%), respectively. These two measures are highly correlated, a correlation of 0.82, which suggests co-movements in the same direction.

We also analyze the news impact curves resulting from the BiN-GARCH model. Panel A of Figure 7 shows the reaction of market, downside, and upside volatilities to return shocks. The asymmetric pattern that emerges for market volatility is interesting and corroborates existing findings.[17] Positive and small negative return shocks lower market volatility. Large negative shocks, on the other hand, significantly increase market volatility in comparison with positive shocks of the same magnitude. This asymmetric pattern transmits to downside and upside volatilities too. Negative or small positive return shocks either do not change or slightly reduce

[17] See Bollerslev, Litvinova, and Tauchen (2006).

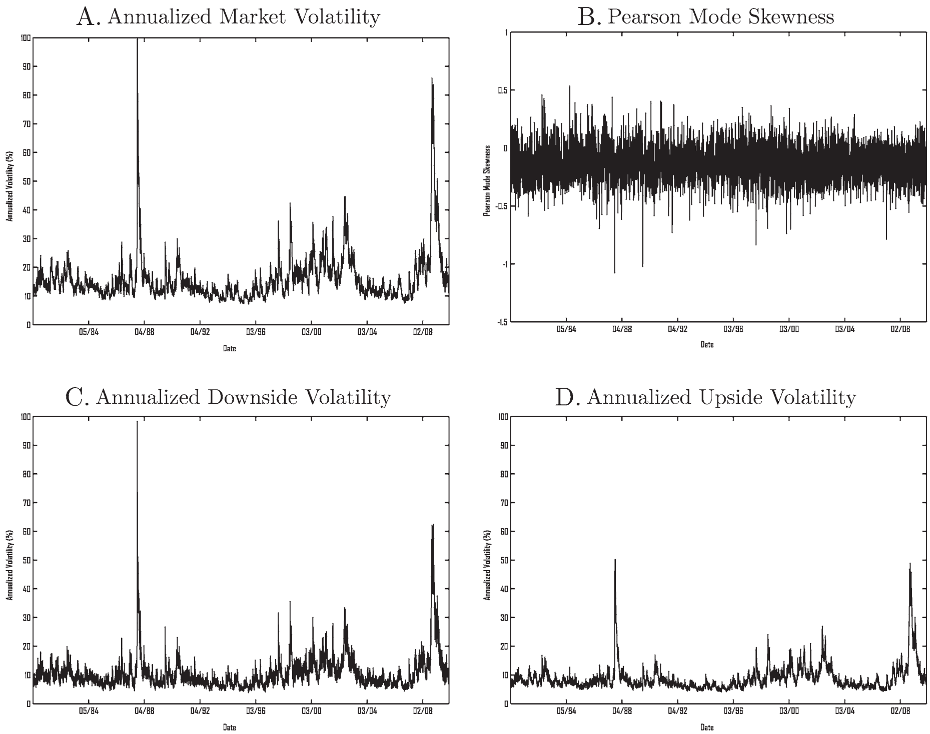

Figure 6. BiN-GARCH conditional moments for S&P 500 annualized daily excess returns
The figures above show conditional moments for annualized daily S&P 500 index excess returns. These values are filtered after fitting the returns series, using Specification (IV) in Table 2. In Panel A, we report conditional "total" market volatility. Conditional Pearson mode skewness appears in Panel B. We report filtered annualized downside and upside market volatility measures in Panels C and D, respectively. Sampling period is January 2 1980 to December 30 2009. Source: Thomson Reuters' Datastream.

upside volatility. On the other hand, positive return shocks increase upside volatility sharply. In contrast, while positive return shocks lower downside volatility, negative return shocks of the same magnitude cause noticeably larger increases in downside volatility.

Finally, Panel B of Figure 7 displays the reaction of market asymmetry to positive and negative return shocks. It is immediately obvious that this response is highly asymmetric and kinked at the origin. While negative return shocks cause linear reductions in market asymmetry, positive return shocks of the same magnitude significantly increase market asymmetry. These increases are arguably nonlinear. We can interpret this pattern as follows: a negative return shock today increases the likelihood of negative return shocks tomorrow. On the other hand, a positive return shock today increases the likelihood of positive returns tomorrow much more than a negative shock of a similar magnitude increases the possibility of

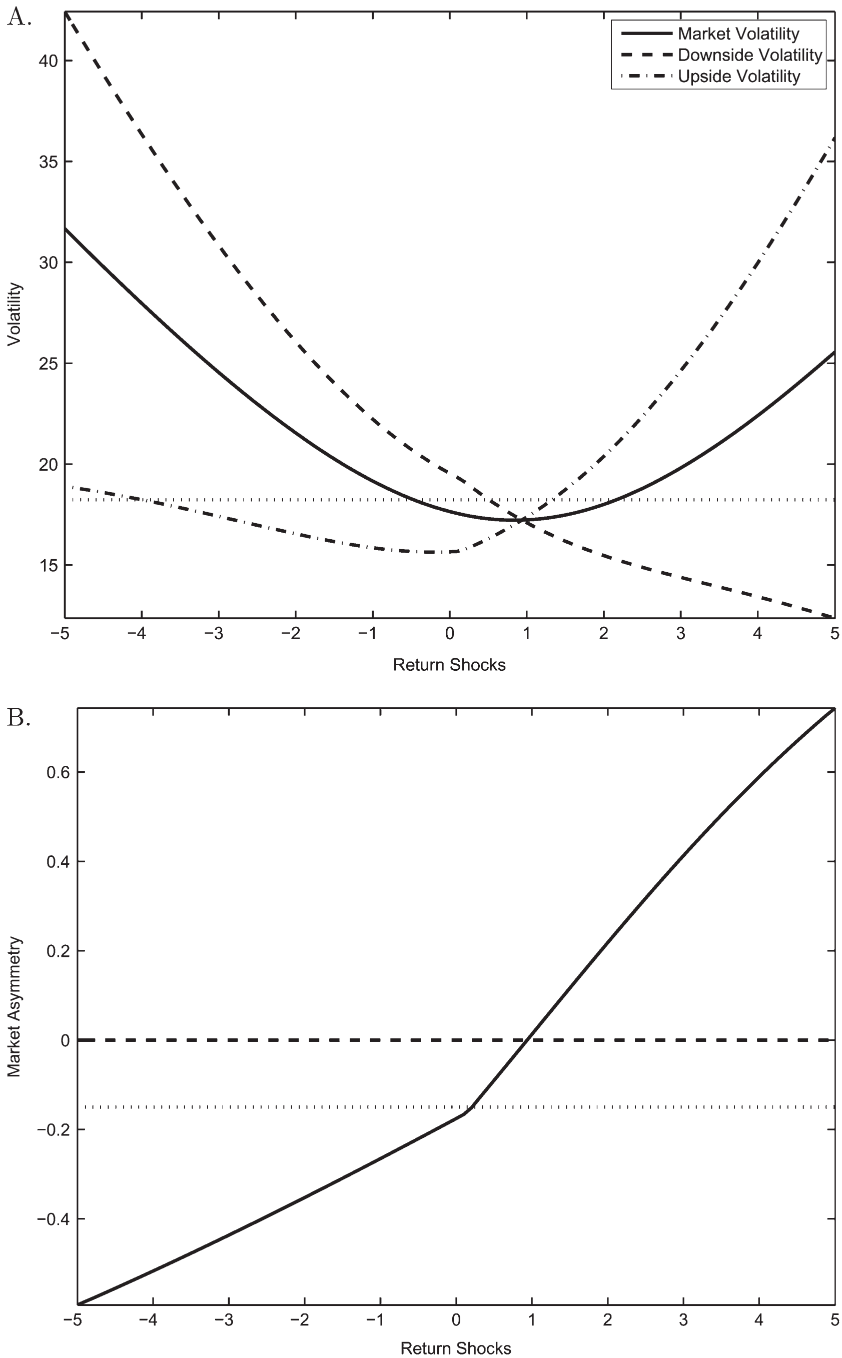

Figure 7. Volatility and asymmetry news impact curves
Panel A of this figure displays the response of market, upside, and downside volatility to negative and positive return shocks. Similarly, in Panel B, we observe the impact of positive and negative return shocks on market asymmetry. We view the response of market asymmetry to negative and positive return shocks as supportive of presence of a "asymmetry in asymmetry" effect in market returns data.

future negative shocks. We find this result to be a nice confirmation of our "asymmetry in asymmetry" assertion.

5.2 EMPIRICAL PERFORMANCE OF BiN-GARCH

So far we have shown that BiN-GARCH model characterizes S&P 500 daily excess returns for 1980–09 period. In this section, we study the performance of restricted and unrestricted BiN-GARCH fitting of S&P 500 returns in the full sample and in three subsamples. As mentioned in the previous section, we use Specification (V) in Table II as the preferred model. Hence we use that specification for estimation in what follows. Estimation results are reported in Tables III and IV.

We break up the S&P 500 excess returns sample into three subsamples: 1980–89, 1990–99, and 2000–09. Each subsample includes significant events or market activity periods. For example, the crash of October 1987 and the oil shock of 1980 are in 1980–89 subsample. The second subsample, 1990–99, includes the dot-com boom of the late 1990s. The last subsample includes data for the Great Recession. We fit the data in each subsample using maximum likelihood methodology and the Specification (IV) discussed in the previous section.

In what follows, two sets of models are estimated, based on our discussion in Sections 2.2 and 3. Table III reports the results from fitting the full BiN-GARCH

Table III. ML Estimation of unrestricted BiN-GARCH model using S&P 500 daily excess returns: robustness checks

This table presents maximum likelihood estimation results for binormal GARCH model using S&P 500 daily excess returns. The sample spans continuously compounded value-weighted returns on the S&P 500 Index from January 2 1980, to December 31 2009, and divided into three subsamples. Standard errors are given in parentheses. *, †, and ‡ denote statistical significance at 1%, 5%, and 10% levels, respectively.

Estimated parameter	1980–89	1990–99	2000–09	1980–09
λ_0	−0.0015 (0.0011)	−0.0004 (0.0005)	−5.09E–05 (0.0004)	−0.0004 (0.0003)
λ_1	0.5829* (0.1107)	0.5633* (0.0660)	1.8367‡ (1.1681)	0.6608* (0.0585)
λ_2	−0.3755† (0.1446)	−0.4308* (0.0747)	−2.1968 (1.5432)	−0.5805* (0.0751)
β_0	4.09E–06* (9.56E–07)	1.25E–06* (3.65E–07)	1.29E–06* (3.43E–07)	1.62E–06* (2.6E–07)
β_1	0.8750* (0.0197)	0.8821* (0.0184)	0.8534* (0.0192)	0.8796* (0.0097)
β_2	0.0600* (0.0095)	0.0571* (0.0096)	0.0471* (0.0078)	0.0621* (0.0050)
θ	0.6196* (0.1443)	0.8858* (0.1348)	1.4136* (0.2071)	0.8546* (0.0777)
α_0	−0.0455‡ (0.0264)	−0.1677* (0.0330)	−0.0879‡ (0.0490)	−0.0815* (0.0197)
α_1	0.1341* (0.0467)	0.3653* (0.0500)	0.0121 (0.0184)	0.1568* (0.0318)
α_2	0.1025* (0.0202)	0.0175 (0.0303)	0.0419 (0.0463)	0.0680* (0.0167)
α_3	0.2632† (0.1080)	0.1164 (0.0801)	0.2796 (0.2166)	0.2842* (0.0737)
LogLik	8,221.70	8,664.72	7,887.08	24,701.69

Table IV. ML estimation of restricted BiN-GARCH model using S&P 500 daily excess returns: robustness checks

This table presents maximum likelihood estimation results for binormal GARCH model using S&P 500 daily excess returns. The sample spans continuously compounded value-weighted returns on the S&P 500 Index from January 2 1980 to December 31 2009, and divided into three subsamples. Standard errors are given in parentheses. *, †, and ‡ denote statistical significance at 1%, 5%, and 10% levels, respectively.

Estimated parameter	1980–89	1990–99	2000–09	1980–09
λ_0	−0.0004 (0.0009)	−0.0003 (0.0005)	−0.0005 (0.0003)	−0.0002 (0.0003)
γ	0.9280* (0.2913)	1.9824* (0.1086)	1.0169* (0.0415)	2.1275* (0.7998)
α	0.8692* (0.2181)	0.8326* (0.1325)	0.9703* (0.0831)	0.9635* (0.3514)
β_0	3.63E–06* (8.535E–07)	1.08E–06* (3.11E–07)	1.63E–06* (3.51E–07)	1.49E–06* (2.37E–07)
β_1	0.8855* (0.0174)	0.8981* (0.0155)	0.8308* (0.0175)	0.8877* (0.0083)
β_2	0.0615* (0.0098)	0.0562* (0.0088)	0.0561* (0.0079)	0.0620* (0.0049)
θ	0.5062* (0.1217)	0.7404* (0.1204)	1.3870* (0.1579)	0.7872* (0.0677)
α_0	0.0029 (0.0269)	−0.1469* (0.0459)	−0.0934 (0.0669)	−0.0560* (0.0191)
α_1	0.0476 (0.0455)	0.2921* (0.0806)	−0.0770 (0.0745)	0.1055* (0.0306)
α_2	0.1443* (0.0307)	0.0340 (0.0335)	0.1198† (0.0526)	0.0854* (0.0171)
α_3	0.3302† (0.0937)	0.1284 (0.1255)	−0.0385 (0.2498)	0.3177* (0.0731)
LogLik	8,218.07	8,651.42	7,882.44	24,698.53

model, presented in Equation (16), to S&P 500 excess returns. Notice that in this step, we do not impose any restrictions on λ_1 and λ_2. The following regularities are observed in this table: First, we observe that once we allow for a dynamic formulation in conditional mode, we get significant variation in estimated parameters across subsamples. Specifically, estimated λ_1 and λ_2 parameters are statistically close in the first two subsamples, but they are significantly larger in the 2000–09 subsample. We attribute this observation to the prolonged turmoil that engulfed financial markets during the 2007–09 period. In contrast, events such as the crash of 1987 typically did not last as long. We believe that this lengthy duration of market instability altered the magnitude of the parameters governing the dynamics of the model, but did not change the fundamental dynamics. Except for estimated λ_2 in 2000–09 subsample, all other reported estimated parameters are statistically significant across the subsamples, and the absolute values of λ_1 and λ_2 are within one standard error of each other within all subsamples and across the first two subsamples.

As expected, estimated NGARCH parameters, β_0 to β_2 and θ, are statistically significant, have the expected signs, and the expected sizes. On the other hand, the picture is more complicated once we study the dynamics of Pearson's mode skewness. All the estimated parameters are statistically significant in 1980–89 and 1980–09 samples. However, most of estimated κs are not statistically significant in 1990–99 and 2000–09 samples. Some of the regularities observed and discussed in Section 5.1 and Table II are broadly observed across the subsamples as well. For example, estimated κ_1s are generally larger than estimated κ_2s, implying asymmetry in asymmetry. Estimated κ_3 parameters, regardless of statistical significance, are roughly the same size, which is substantially less than 1. This observation implies low persistence in Pearson's mode skewness.

In the next step, we recover the structural parameters of DA preferences, by imposing the restrictions obtained from Equation (11) on λ_1 and λ_2.[18] We substitute for λ_1 and λ_2 in Equation (16) from the analytical solutions in Equation (11). We report the estimation results in Table IV. It is immediately obvious that NGARCH estimated parameters are statistically significant, regardless of the sample used in estimation. Estimated parameters for Pearson's mode skewness are all significant for the full sample, but their statistical significance weakens in subsamples. There is also considerable variation in size and signs of these parameters across subsamples. But even with all these concerns, we still observe substantial asymmetry in asymmetry in Pearson's mode skewness dynamics.

[18] Equation (11) explicitly links λ_1 and λ_2 to the disappointment aversion parameter, α, and the coefficient of risk aversion, γ, which are the structural parameters in DA preferences. As noted earlier, λ_1 and λ_2 do not depend on δ.

Estimating the restricted model delivers two important contributions. First, it allows us to recover the structural parameters of DA preferences. We observe the following in our estimates: First, all estimated structural parameters, α and γ, are statistically significant. Second, there is variation in our estimates across subsamples and the full sample. This variation is more pronounced in estimates of the coefficient of risk aversion, γ. Estimated γs range between 0.9289 in 1980–89 subsample to 2.1275 for the full sample. These values are within the (0, 10) range that is generally considered acceptable in the literature; see Mehra and Prescott (1985). The range of variation is more limited in the estimated disappointment aversion parameter, α. We observe a range between 0.8326 in 1990–99 to 0.9703 in 2000–09 period. These point estimates are all less than one, which is the main requirement for DA preferences. Estimated values of α based on experimental data in Choi, Fisman, Gale, and Kariv (2007) imply values in 0.70 to 0.98 range.[19] Third, our estimates imply modest, but persistent, disappointment aversion. The interesting observation here is the time variation in disappointment aversion and risk aversion parameters. We will study this issue in future research.

Second, estimating the restricted model allows us to extract the market price of risk, implied by time-varying risk–return trade-off defined in Equation (17). The time-varying market price of risk process is characterized by Equation (13). We are also interested in the relationship between p_t and λ_t^*. The top panel in Figure 8 reports the time series of annualized daily values for the market price of risk, computed for S&P 500 excess returns, based on Equation (13). It is immediately clear that market price of risk is positive. Thus, our findings support the results of Ghysels, Santa-Clara, and Valkanov (2005) and Ludvigson and Ng (2007). However, we show that the market price of risk is a process with considerable time variation. The average annualized λ_t^* is 0.4401, close to the annual Sharpe ratio computed from US historical data. The bottom panel of Figure 8 shows the relationship between the market price of risk and Pearson mode skewness. The sample correlation between λ_t^* and p_t is 0.6435, which implies that the market price of risk decreases when market asymmetry decreases. Logically, if extreme negative returns are more likely, an investor would not demand more risk, and this fall in the demand of risk induces a lower market price of risk. The top panel in this figure shows that the market price of risk is lower during recessions in our sample. Besides, the bottom panel of the figure showcases asymmetry in asymmetry of p_t process mentioned in the discussion of Equation (15). The sample contains more frequent large negative realizations of p_t than positive realizations.

[19] Choi, Fisman, Gale, and Kariv (2007) use a static formulation for DA preferences and define disappointment aversion parameter in $(1, +\infty)$ range. We transform their estimated values to the (0, 1) range studied in this paper. Our specification corresponds to Bonomo *et al.* (2011). They also define the disappointment aversion parameter so that it belongs to (0, 1).

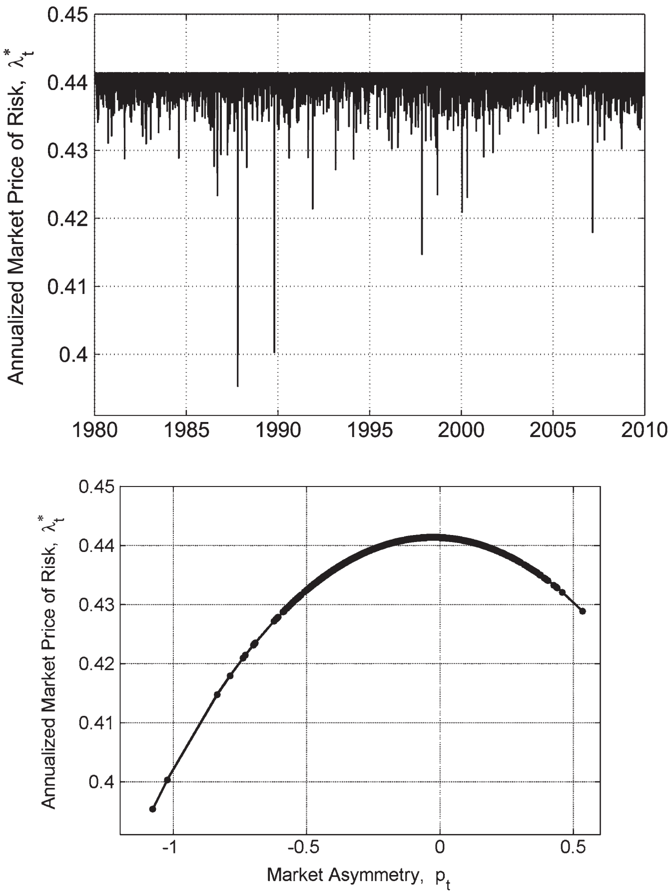

Figure 8. Relative downside volatility, conditional skewness, and market price of risk
The top panel of this figure displays the time series of market price of risk filtered from S&P 500 data at daily frequency for January 1980 to December 2009. These values are computed based on $\lambda_t^* = (1-(\lambda_1-\lambda_2)\sqrt{\pi/8})p_t + (\lambda_1-\lambda_2)\sqrt{1-(3\pi/8-1)p_t^2}$ equation. The bottom panel in this figure displays the relationship between λ_t^* and p_t filtered from S&P 500 data at daily frequency for January 1980 to December 2009.

5.3 BiN-GARCH AND INTERNATIONAL DATA

In the next step, to verify the ability of BiN-GARCH to characterize the excess return and risk–return trade-off dynamics beyond S&P 500 index, we model daily MSCI index excess returns for five major international financial markets using BiN-GARCH. The sample includes Australia, Germany, Japan, Switzerland, and UK. Thus, it includes members of the Euro zone, representatives from Asia and Oceania, and two important European markets, Switzerland and the UK, which are not Euro members.[20] As mentioned in Section 4, we use MSCI country index values downloaded from *Thomson Reuter's Datastream*. We concurrently estimate conditional mode, variance, structural parameters, and asymmetry using maximum likelihood methodology.

Panel A in Table V reports the results from fitting the international data using the full BiN-GARCH model, presented in Equation (16). Similar to our results reported in Section 5.2, statistical evidence for existence of a positive linear relationship between both the conditional mode and downside risk and upside uncertainty is quite strong. Moreover, estimated parameters have the expected sign, positive λ_1 and negative λ_2, and their absolute values are generally within one standard error from each other. These regularities are true across the board for the full sample of 1980–09. These regularities broadly hold within and across the three subsamples of 1980–89, 1990–99, and 2000–09. However, the results within subsamples are not as strong as the results for the full sample. This may be due to using shorter series for subsamples. We find these results to indicate significant statistical support for our model and estimation procedure in the international data.

Panel B in Table V reports the results from fitting the international data, using the method used in obtaining the results presented in Table IV. Again, our goal is to recover the structural parameters of the DA preferences, by imposing the restrictions implied by Equation (11) and the analytical solutions linking λ_1 and λ_2 to the structural parameters γ and α. Similar to what we see in Sections 5.1 and 5.2, we observe statistically significant estimates for γ and α in the full sample, and reasonable statistical support within and across the subsamples. There is more variation in estimated values of γ in comparison with estimated values of α. Based on our findings, the Japanese investor is the most disappointment averse. The full-sample estimated α for Japan is 0.8526, which is considerably lower than 0.9032 observed for Switzerland, the second lowest value. Our estimates also suggest that the level of risk aversion, implied by estimated values of γ, is considerably higher in the USA compared with the other markets studied here. Estimated γ for the USA is 2.1275 in the full sample. The highest value of γ in the international data is 1.0191 for Switzerland in the same time period.

Figure 9 shows the behavior of the risk aversion-disappointment aversion coefficient couple, (γ, α), in twenty-two financial markets, and over the three time periods. We estimate these parameters by fitting the restricted version of BiN-GARCH to MSCI data for the Organisation for Economic Co-operation and Development countries and S&P 500 data. We do not report all the estimation results to save space; they

[20] In the online appendix, we provide empirical evidence in support of successful fitting of the data and positive BiN-GARCH risk–return tradeoff for 10 financial markets. Since they are generally statistically significant and show very little variation across sub-samples, we do not report the estimated parameters for the volatility process and Pearson mode skewness. Data source is Thomson Reuters Datastream and MSCI index data.

Table V. Restricted and unrestricted ML estimation of the BiN GARCH model using international data

This table reports maximum likelihood estimation results of the BiN-GARCH model for five international markets' daily excess returns. The sample includes continuously compounded value-weighted returns on country indexes starting on January 1, 1980 and ending in December 31 2009. Standard errors are given in parentheses. *, †, and ‡ indicate statistical significance of the estimated parameters at 1%, 5%, and 10% levels, respectively. λ_i are as in $m_t = \lambda_0 + \lambda_1\sigma_{1,\,t} + \lambda_2\sigma_{2,\,t}$. γ and α are derived by analytically solving Equation (11), and denote coefficients of risk aversion and coefficients of disappointment aversion, respectively. Source: Thomson Reuters Datastream.

	Panel A: Unrestricted model					Panel B: Restricted model				
Country	Parameter	1980–88	1990–99	2000–09	1980–09	Parameter	1980–88	1990–99	2000–09	1980–09
Australia	λ_0	0.0025† (0.0012)	−0.0005 (0.0013)	1.28E–05 (0.0011)	0.0005 (0.0005)	λ_0	0.0005‡ (0.0003)	−0.0003 (0.0012)	0.0005† (0.0002)	9.16E–05 (0.0002)
	λ_1	0.2130† (0.0856)	0.7797* (0.0966)	0.5252* (0.0865)	0.5012* (0.0553)	γ	0.0008 (0.0454)	1.0019* (0.0158)	0.0002 (0.0170)	0.9660* (0.0226)
	λ_2	−0.3366* (0.0068)	−0.7264* (0.1604)	−0.4294†(0.1864)	−0.4842* (0.0699)	α	0.9998* (0.0016)	0.9377* (0.2696)	0.9999* (0.0020)	0.9750* (0.0324)
Log Lik		6,718.80	9,093.35	7,744.29	23,464.38		6,676.80	9,093.28	7,736.85	23,441.19
Germany	λ_0	−0.0020 (0.0012)	−0.0003 (0.0007)	3.80E–04 (0.0006)	−0.0002 (0.0005)	λ_0	−0.0018 (0.0011)	−0.0007 (0.0006)	0.0002 (0.0003)	−0.0002 (0.0004)
	λ_1	0.4920* (0.1533)	0.9006* (0.0514)	1.0338* (0.1253)	0.7959* (0.0989)	γ	10.1301 (41.2002)	0.1186 (3.7939)	0.1406 (0.9781)	1.0150* (0.2286)
	λ_2	−0.3384† (0.1375)	−0.8785* (0.0803)	−1.1167* (0.1707)	−0.7676* (0.1321)	α	0.9018 (1.1654)	0.7856* (0.1492)	0.9998* (0.0091)	0.9433* (0.0911)
Log Lik		6,985.08	8,863.94	7,454.72	23,256.33		6,978.24	8,862.44	7,450.94	23,256.32
Japan	λ_0	−0.0037* (0.0011)	−0.0018† (0.0007)	−0.0001 (0.0012)	−0.0010‡ (0.0005)	λ_0	−0.0010 (0.0009)	−0.0017† (0.0007)	−0.0016 (0.0010)	−0.0008 (0.0005)
	λ_1	0.7194* (0.0379)	0.7976* (0.1447)	1.2279* (0.2444)	0.4692* (0.0876)	γ	11.0943 (21.7951)	1.0411 (0.7848)	1.0640* (0.3385)	0.9847* (0.0614)
	λ_2	−0.3060* (0.0898)	−0.6874* (0.1480)	−1.3102* (0.3149)	−0.3903* (0.0895)	α	0.9012 (0.5814)	0.7666* (0.1100)	0.8034* (0.1577)	0.8526* (0.0874)
Log Lik		7,161.26	8,482.08	7,499.13	23,067.50		7,139.41	8,482.00	7,496.04	23,057.95
Switzerland	λ_0	−0.0003 (0.0012)	−0.0018 (0.0011)	3.58E–05 (0.0006)	−0.0006 (0.0005)	λ_0	−4.69E–05 (0.0009)	−0.0009 (0.0009)	−4.75E–05 (0.0003)	−0.0003 (0.0004)
	λ_1	0.3517† (0.1671)	0.6313* (0.0915)	0.8828* (0.2605)	0.5543* (0.0786)	γ	1.0065 (19.6501)	0.9684* (0.0663)	0.9897* (0.1774)	1.0191* (0.0693)
	λ_2	−0.3219‡ (0.1699)	−0.3655* (0.1222)	−0.8905* (0.3060)	−0.4466* (0.0910)	α	0.9915‡ (0.5463)	0.7431* (0.1709)	0.9785* (0.0195)	0.9032* (0.0943)
Log Lik		7,302.69	9,249.61	8,092.65	24,611.15		7,296.83	9,243.38	8,092.59	24,603.92
UK	λ_0	0.0005 (0.0020)	−0.0004 (0.0009)	0.0006 (0.0005)	−0.0001 (0.0004)	λ_0	8.88E–05 (0.0007)	−8.54E–05 (0.0008)	−7.77E–07 (0.0002)	−0.0001 (0.0003)
	λ_1	0.3826‡ (0.2340)	0.5757* (0.0987)	1.5151 (1.3013)	0.5005* (0.1534)	γ	0.6458 (3.3655)	0.8501 (0.5580)	0.0110 (0.2816)	1.0015* (0.0062)
	λ_2	−0.3009 (0.3287)	−0.4818* (0.1157)	−1.7332 (1.5740)	−0.4320† (0.1822)	α	0.9849* (0.0885)	0.9287* (0.2082)	0.9996 (0.0112)	0.9577* (0.0690)
Log Lik		6,867.49	9,474.77	7,987.76	24,254.38		6,863.80	9,468.09	7,983.64	24,252.54

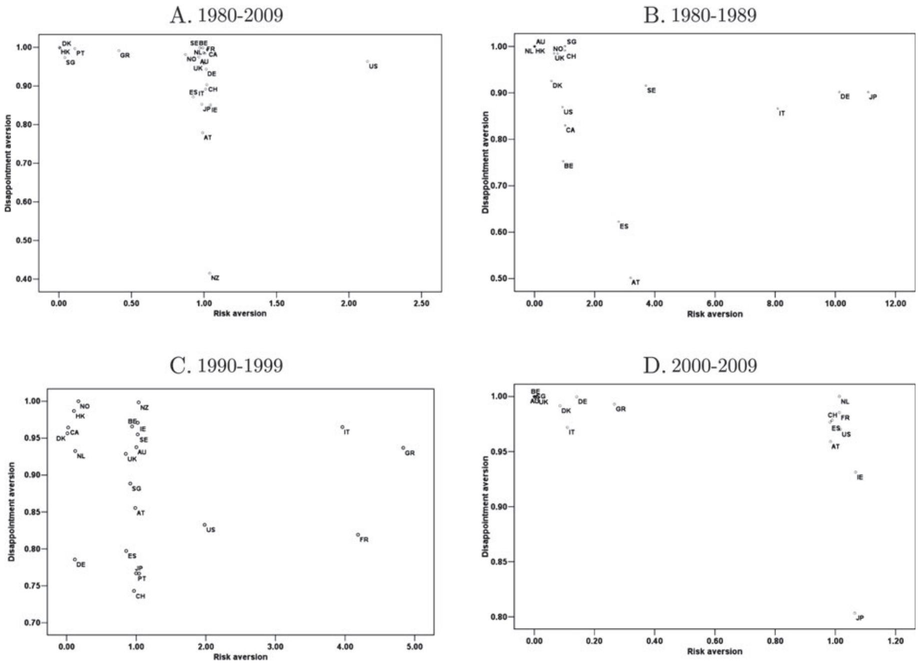

Figure 9. Estimated values of coefficients of risk aversion and disappointment aversion across markets This figure plots the estimated pairs of coefficients of risk aversion (γ, on the horizontal axis) and disappointment aversion (α, on the vertical axis) in various markets and across different time periods. These estimated values are obtained by fitting the data to the restricted BiN-GARCH model, based on the analytical solution to Equation (12). Source: Datastream and MSCI. Country codes: Australia (AU), Austria (AT), Belgium (BE), Canada (CA), Denmark (DK), France (FR), Germany (DE), Greece (GR), Hongkong (HK), Ireland (IE), Italy (IT), Japan (JP), the Netherlands (NL), New Zealand (NZ), Norway (NO), Portugal (PT), Singapore (SG), Spain (ES), Sweden (SE), Switzerland (CH), United Kingdom (UK), United State (US).

are available upon request. It is immediately clear that these coefficients vary across subsamples and over time. For example, γ drops in Italy, Germany, and Japan since 1980s, and it increases in the USA over the same period. Also, the DA coefficients, α, on average, have increased since the 1980–89 period. The dynamics of these coefficients are interesting. We will also address this issue in future research.

To summarize, empirical evidence from the international data support modest disappointment aversion and variation in risk aversion across countries and across time. International evidence lends strong statistical support for a positive impact of downside volatility and a negative impact of upside uncertainty (in other words, positive impact of relative downside risk) on the conditional mode in the BiN-GARCH model. Thus, we claim that an empirical study of risk–return tradeoff necessitates a study of downside risk, time-varying skewness, and modeling the price of risk as a process. We believe that our candidate in Equation (13) provides a good starting point for future research.

5.4 DOWNSIDE VOLATILITY AND HIGH FREQUENCY DATA

At this point, the reader may question how much our results, both theoretical and empirical, depend on the parametric statistical model developed and used in the paper. We claim, and provide extensive empirical evidence to support this claim, that our findings are not dependent on the parametric model used for estimation. Another legitimate question is whether relative downside risk is in fact a proxy for other known pricing factors. One such factor that has attracted considerable attention in recent years in the "variance premium."[21] In this section, we present empirical evidence in support of the claim that relative downside risk is not a proxy for variance premium (or other similar quantities), but an independent, and one may even say complementary factor to the variance premium.

We achieve these goals by departing from the parametric framework used so far in the paper. Instead, we build nonparametric measures of relative downside risk based on high frequency data for returns. Since one needs good quality high frequency data to carry out what follows, we limit the study of downside volatility and high frequency data to S&P 500 returns and 1986–09 time period. The most important of risk measures in this class is the realized volatility, which provides an *ex post* measure of volatility. Following Bollerslev, Tauchen, and Zhou (2009), we construct model-free realized volatility measures, as opposed to options implied volatilities of Black–Scholes. Many studies in finance and econometrics are devoted to realized volatility. Among them, we note Andersen, Bollerslev, Diebold, and Labys (2001), Andersen, Bollerslev, Diebold, and Labys (2003), Bollerslev and Zhou (2006), and Bollerslev, Tauchen, and Zhou (2009). This list is by no means exhaustive.

We construct our measures following the common practice in the realized volatility literature by summing up finely sampled squared return realizations over a fixed time interval,

$$\mathrm{RV}_t = \sum_{j=1}^{n_t} r_{j,t}^2, \tag{19}$$

[21] Variance premium is the difference between "model-free" implied and realized variances; see Bollerslev, Tauchen, and Zhou (2009).

Table VI. Evaluating the conditional relative downside variance: mincer-zarnowitz regressions

This table reports the results of running Mincer and Zarnowitz (1969) regressions for realized downside volatility against conditional downside volatility: $\mathrm{RV}^d_{t+1}(h) - \mathrm{RV}^u_{t+1}(h) = \varphi_0 + \varphi_1(\mathrm{Var}^d_t[R_{t+1}] - \mathrm{Var}^u_t[R_{t+1}]$ for 1986–09 S&P 500 data. h denotes sampling frequency in minutes. Standard errors are reported in parentheses. *represents statistical significance at 5% or better confidence level. Source: Thomson Reuters Datastream.

h	5	10	15	20	30	60
ϕ_0	−4.50E−05*	−3.36E−05*	−1.49E−05*	−6.42E−06	−3.75E−06	1.20E−06
	(4.29E−06)	(3.43E−06)	(2.76E−06)	(2.27E−06)	(2.05E−06)	(1.98E−06)
ϕ_1	2.631*	2.235*	1.583*	1.286*	1.199*	1.035*
	(0.045)	(0.036)	(0.029)	(0.024)	(0.021)	(0.020)
R^2	0.370	0.400	0.340	0.331	0.340	0.300

where there are n_t high frequency returns in period t, and $r_{j,t}$ is the the jth high frequency return in period t. We also construct the realized downside and upside variance series as

$$\mathrm{RV}^d_t = \frac{n_t}{n^d_t}\sum_{j=1}^{n_t} r^2_{j,t} I\left(r_{j,t} < \bar{m}_t\right), \quad \mathrm{RV}^u_t = \frac{n_t}{n^u_t}\sum_{j=1}^{n_t} r^2_{j,t}\left(r_{j,t} \geq \bar{m}_t\right), \qquad (20)$$

where n^d_t and n^u_t are, respectively, the number of high frequency returns below and above the mode of return, $\bar{m}_t$, in period t, and where $I(\cdot)$ denotes the indicator function. Following Bollerslev, Tauchen, and Zhou (2009), we construct realized upside and downside variance measures for daily, monthly, and quarterly periods.

In the first step, we study the ability of our parametric downside risk measure to forecast the realized relative downside risk. This step is motivated by Andersen and Bollerslev (1998). In particular, they show that since $\mathbb{E}_t(\mathrm{RV}_{t+1}) \approx \mathrm{Var}_t(R_{t+1})$, realized volatility provides an easy to evaluate measure of return volatility through Mincer and Zarnowitz (1969) regressions. We use the same logic to develop an evaluation framework for realized measures, constructed following Equation (20) as forecasts for upside and downside volatility, using Mincer–Zarnowitz regressions. Formally, we fit[22]

$$\mathrm{RV}^d_{t+1}(h) - \mathrm{RV}^u_{t+1}(h) = \varphi_0 + \varphi_1(\mathrm{Var}^d_t[r_{t+1}] - \mathrm{Var}^u_t[r_{t+1}]) + \epsilon_{t+1}. \quad (21)$$

The Mincer–Zarnowitz framework implies that if the right-hand side variable is a good predictor of the left-hand side variable, then $\phi_0 = 0$ and $\phi_1 = 1$. The empirical results for S&P 500 data are presented in Table 6. The following is immediately obvious: the intercept parameters become less statistically significant as we increase the sampling frequency. This result is not surprising. In 5- to 10-minute

22 $Var^d_t[r_{t+1}]$ and $Var^u_t[r_{t+1}]$ are defined by Equation (5).

frequency, data may be contaminated by market microstructure noise. As the frequency of sampling decreases, the importance of microstructure noise declines, and we observe more "efficiency" in the Mincer–Zarnowitz sense. Similarly, while all slope parameters are statistically significantly different from zero, we cannot reject the null hypothesis of $\phi = 1$ for small values of h. Thus, these regressions imply that for data sampled at moderate high frequencies such as 20, 30, or 60 minutes, our parametric model is an almost perfect predictor for nonparametric *ex post* realized downside risk measures. Given the reasonably high R^2 values of these regressions, we can confidently claim that our parametric measure explains at least one-third of the variation in the nonparametric realized downside volatility.

This, in our opinion, is a significant contribution to the empirical asset pricing literature. Since the work of Hansen (1994), many studies have tried to propose a model for time varying conditional skewness. However, a testing procedure to evaluate the validity of competing specifications proved to be elusive. We believe that we provide this crucial tool since relative downside risk is driven by conditional skewness. Testing for validity of Equation (21) is in fact a test for validity of our specification of conditional skewness.

Next, we empirically study the relationship between (realized) relative downside volatility and the variance premium. We use the formulation and methodology introduced by Bollerslev, Tauchen, and Zhou (2009) for computing the variance premium in S&P 500 returns. We then fit the following models to returns:

$$\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{RV}_t + \tilde{\lambda}_2 \mathrm{RDV}_t, \tag{22}$$

$$\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{RV}_t + \tilde{\lambda}_3 \mathrm{VRP}_t, \tag{23}$$

$$\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{RV}_t + \tilde{\lambda}_2 \mathrm{RDV}_t + \tilde{\lambda}_3 \mathrm{VRP}_t, \tag{24}$$

where $\mathbb{E}_t(r_{t+1}) - r_{f,t+1}$ represents excess returns over the 3-month Treasury Bill rate, RV_t is defined in Equation (19), RDV_t is the nonparametric measure of relative downside risk defined earlier, and VRP_t is the variance risk premium.

We refer to Equations (22–24) as Model I to Model III in Table VII, where we present the empirical results. The table shows that regardless of the model, the intercept parameter, $\tilde{\lambda}_0$ is generally not statistically significant. The estimated coefficient for relative downside volatility, $\tilde{\lambda}_2$, is statistically significant and positive in both Models I and III. The estimated variance premium coefficient is slightly less stable. The expected positive sign turns negative at the quarterly level in both Model II and III. The realized variance coefficient also demonstrates a similar pattern. At the quarterly level, the realized variance coefficient is statistically significant for Models I and III but has a negative sign for all three models. This problem is limited to quarterly regressions and is not present for the other regressions.

Table VII. Relative downside volatility and the variance premium

This table presents the empirical results for fitting S&P 500 returns to the following three models. Model I: $\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{Var}_t[R_{t+1}] + \tilde{\lambda}_2 \mathrm{RDV}_t$, Model II: $\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{Var}_t[R_{t+1}] + \tilde{\lambda}_3 \mathrm{VRP}_t$, and Model III: $\mathbb{E}_t(r_{t+1}) - r_{f,t+1} = \tilde{\lambda}_0 + \tilde{\lambda}_1 \mathrm{Var}_t[R_{t+1}] + \tilde{\lambda}_2 \mathrm{RDV}_t + \tilde{\lambda}_3 \mathrm{VRP}_t$, where $r_{f,t}$ is the 3-month T-Bill rate, RDV_t is the relative realized downside volatility, and VRP_t is the variance risk premium, defined and computed as in Bollerslev, Tauchen, and Zhou (2009). $\bar{R}^2$ is the adjusted R^2.

	Model I			Model II			Model III		
Parameter	Daily	Monthly	Quarterly	Daily	Monthly	Quarterly	Daily	Monthly	Quarterly
$\tilde{\lambda}_0$	−3.272E–04 (2.325E–04)	0.0062 (0.0017)	0.0051 (0.0083)	−0.0008 (0.0005)	0.0007 (0.0026)	−0.0205† (0.0073)	−0.0007 (0.0006)	0.0056‡ (0.0026)	0.0199 (0.0107)
$\tilde{\lambda}_1$	1.9523 (1.2666)	3.6947‡ (1.9107)	−5.4935† (2.0375)	7.5008* (1.2025)	6.0802* (1.7890)	−0.2752 (1.2048)	5.3101* (1.3267)	3.5880‡ (1.9449)	−5.3509* (2.0134)
$\tilde{\lambda}_2$	7.3190* (0.8913)	10.1856* (0.8559)	5.4750* (0.7770)				6.8169* (0.8914)	10.1188* (0.8852)	5.8184* (0.7565)
$\tilde{\lambda}_3$				4.8378* (0.5340)	4.5465* (1.2873)	−0.0671 (1.0284)	4.4239* (0.5468)	0.4050 (1.3679)	−2.7630† (1.2248)
$\bar{R}^2$	0.0172	0.0285	0.0751	0.0096	0.0270	0.0528	0.0175	0.0344	0.1299

Another interesting pattern is discernible in the reported adjusted R^2 measures ($\bar{R}^2$) in Table VII. They increase, regardless of the model, once we move from daily to quarterly frequency, which is not surprising. On the one hand, we notice that R^2s for Model I are larger than $\bar{R}^2$s s from Model II at the same frequency, for instance, the daily $\bar{R}^2$ for Model I is 0.0172, while Model II has $\bar{R}^2$ equal to 0.0096, and for monthly frequency, $\bar{R}^2$ for Model I is 0.0285 and $\bar{R}^2$ for Model II is 0.0270. This may mean that RDV_t has more power in explaining the variations in excess returns compared to VRP_t in Model II. On the other hand, reported $\bar{R}^2$ for Model III is very close to the sum of reported $\bar{R}^2$s for Models I and II. This observation is particularly visible at the quarterly level. We interpret this observation as evidence against the usual increase in $\bar{R}^2$ accompanied by adding more regressors, and as evidence in favor of the power of relative downside risk as a pricing factor. Adding the variance premium factor to Model I does very little in terms of changing the size and sign of estimated $\tilde{\lambda}_2$s s in Model III. We interpret this observation as evidence against the assertion that relative downside volatility might be a proxy for the variance premium.

6. Concluding Remarks

In this paper, we introduce a new methodology for assessment and study of the risk–return trade-off in S&P 500 excess returns and in international equity markets. We propose a discrete-time dynamic model of asset prices with binormal return innovations: the BiN-GARCH model, which nests the canonical NGARCH model. Using an intuitive endowment and representative agent equilibrium model, we show that demand for relative downside risk compensation arises in familiar theoretical settings with only the assumption that the agent distinguishes between downside and upside risk in the market. We then test our theoretical model using annualized daily index excess returns from five international equity markets and the S&P 500 index.

Our study suggests strong empirical support for the four main assertions in this paper. We find that: First, there exists relative downside risk in equity markets, which is compensated through an increase in the conditional mode of returns. Second, the relationship between relative downside risk and the conditional mode is positive. Third, we establish that conditional skewness is a priced factor. Fourth, we find evidence suggestive of time variation in structural parameters of disappointment aversion preferences. Furthermore, our empirical results support a positive value for the time-varying market price of risk in the markets studied. This last result is due to the characteristics of the estimated volatility spillover parameters in the conditional mode and conditional market asymmetry measures. In this sense, our findings provide additional support for studies such as Ghysels, Santa-Clara, and Valkanov (2005), Ludvigson and Ng (2007), and particularly, Rossi and Timmermann (2009).

We have not visited the existence of common or country-specific factors across markets that influence risk premia. Neither have we investigated the existence, differences, or factors affecting time variation of effective risk aversion across countries. We will address these questions in our future research.

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