# Portfolio Optimization and Asset Pricing Implications under Returns Non-Normality Concerns

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**Abstract**

We investigate the implications of non-normality for asset allocation and pricing. Asset returns non-normality is captured through a multivariate normal-exponential model; we develop an estimation procedure based on a generalized method of moments. Investors' non-normality concerns are introduced by adding a linear non-normality constraint to an otherwise standard mean-variance framework. The optimal portfolio solution is obtained in closed form and can be reformulated as a three-fund separation strategy. Suboptimal portfolios that ignore non-normality or are naive in terms of diversification may result in important welfare costs as measured by the certainty equivalent, notably for the most risk-tolerant investors who target large non-normality ratios. In equilibrium, expected returns admit a two-beta representation in which the most important beta in explaining their cross-sectional variation is the one capturing non-normality (more than 60%) while the CAPM beta explains less than 12%.

**Keywords:** Endogenous preference parameters, non-participation, efficient frontier

**JEL Classification:** C130, G110, G120

# 1 Introduction

The distribution of asset returns very often deviates from normality. Two of the common deviations observed are the asymmetry of returns, which means that returns are oddly distributed around their average, and their fat-tailedness, which means that extreme returns are more likely than their prediction by the normal distribution. These important features of return distribution were first ignored in the modern portfolio theory pioneered by Markowitz (1952) but are now accounted for in the most recent portfolio choice literature. The skewness and excess kurtosis of returns distribution, i.e., key ingredients of the returns non-normality, generate a certain amount of downside risk or upside potential that investors might dislike or appreciate while building their portfolios. Therefore accounting for the non-normality of the returns in portfolio allocation is useful when dealing with investment decisions that better characterize investor preferences.

In this article, we extend the standard mean-variance optimization problem to account for returns non-normality and their implied concerns by the investor. The objective of the investor is to minimize portfolio variance by targeting a minimum level of portfolio expected return and a given degree of portfolio non-normality. The proposed theoretical setup is simple and parsimonious as it operates in a static setting, explicitly ruling out any effect that might otherwise arise from purely dynamic channels.

In our framework, we make the strong yet reasonable assumption that risky asset returns in the economy follow an independent and identically distributed (IID) multivariate normal-exponential model. This model is part of the multivariate families developed around the skew-normal distribution (Azzalini; 1985, 1986, 2005, 2020) and is a limiting case of the multivariate extended skew-normal distribution (Adcock and Shutes; 2012). Out of this literature, this article is the first to develop a consistent estimation procedure for model parameters, based on the generalized method of moments (GMM). Without loss of generality, this simple setting allows us to derive key moments of asset returns analytically, which are useful for the estimation of model parameters using a GMM with exact moment conditions. We demonstrate that the proposed model captures well higher-order moments of individual asset returns and higher-order co-moments between assets. Moreover, as illustrated by Dahlquist et al. (2016), the IID multivariate normal exponential model is able to match other key features of the return data, like asymmetric correlations as studied empirically by Ang and Chen (2002) and Hong et al. (2007).

In the multivariate normal-exponential model of asset returns, idiosyncratic security risks follow a joint multivariate gaussian distribution, whereas non-normality is generated by a single common shock that follows an exponential distribution, but upon which different securities have different loadings. These loadings, collected in a parameter vector $a$, measure the degrees of non-normality of the different assets. The sign of the element $a_i$ and its magnitude reveal the sign of the skewness and the joint size of the higher moments of asset $i$, respectively. In fact, because the $k$th cumulant of asset $i$ is $\propto a_i^k$ for $k \geq 3$ under the multivariate normal-exponential model, it means that $a_i$ can be viewed as the level of non-normality of asset $i$ because asset $i$ would be normally distributed if and only if $a_i = 0$. In this universe, we define a non-normality operator that assigns to each asset its loading on the common exponential shock. This operator is linear, just like the expectation operator, a feature that proves useful for generalizing the analytical tools and preserving the mathematical elegance of the mean-variance framework in our portfolio optimization problem. We further define the non-normality ratio as the degree of non-normality $a_i$ divided by the asset risk premium.

Dahlquist et al. (2016) show that if asset returns follow a multivariate normal-exponential model and investors have generalized disappointment aversion preferences of Routledge and Zin (2010), their setting implies a mean-variance-asymmetry optimization problem that is nested into our mean-variance-non-normality framework. As demonstrated by Dahlquist et al. (2016), this simple static setting generates portfolio choice behaviors consistent with real-life situations as well as asset demands consistent with portfolio recommendations by popular financial advisors. Similar to Dahlquist et al. (2016), our setup leads to a three-fund separation strategy: the investor allocates wealth to the risk-free asset and standard mean-variance efficient fund, and to an additional fund reflecting returns non-normality. The optimal portfolio is characterized by the investor's endogenous effective risk tolerance coefficient and the investor's endogenous non-normality concern coefficient. Likewise, in a portfolio choice model where investors' attitude toward risk is captured through the disappointment aversion preferences of Gul (1991), Ang et al. (2005) show that optimal non-participation in risky security markets occurs when the degree of disappointment aversion is large enough, i.e., when it exceeds a certain threshold. In our setting, we show that optimal non-participation occurs for investors who target a normally distributed portfolio with a nonpositive risk premium. The other investors still buy or sell the non-normality-variance portfolio when the targeted risk premium is negative.

A common approach in the literature for studying the portfolio choice implications of non-normal returns is to use a third- or fourth-order Taylor expansion of a differentiable utility function. Non-normality concerns of the investor are then captured by the coefficients of the third- and fourth-order terms in the expansion. These coefficients are set to values implied by some standard utility (as in Jondeau and Rockinger; 2006, Guidolin and Timmermann; 2008, or Martellini and Ziemann; 2010), or freely determined in an ad hoc way (as in Harvey et al.; 2010). In our approach, non-normality concerns arise as a result of adding a linear non-normality constraint to the standard mean-variance portfolio optimization problem. Here, the optimal portfolio strategy, the effective risk tolerance, and the non-normality concern coefficients which we explicitly characterize, are endogenous and all fully depend on the asset menu, the targeted minimum level of portfolio expected return and the given degree of portfolio non-normality.

Multivariate families featuring non-normality of the same kind as the multivariate normal-exponential model have been used to analyze the effect of non-normality, notably skewness, in portfolio choice problems. We relate to Das and Uppal (2004) and Dahlquist et al. (2016) through the assumed asset return distribution, but differ in that our investor preferences are not explicitly and subjectively characterized by a utility function; rather, they are objectively captured by the targeted minimum expected return and degree of non-normality that the investor wants to achieve through her optimal portfolio choice. Thus, our setup is closely related to Simaan (1993). We derive the certainty equivalent summarizing our investor's preferences, which we use to quantify the cost of ignoring returns non-normality as in Das and Uppal (2004) and Dahlquist et al. (2016). We additionally quantify the cost of ignoring non-normality concerns, a situation that may happen, for example, in decentralized investment management due to the mismatch of preferences between a chief financial officer (CFO) who may care about non-normality and his asset managers who may not.[1] Likewise, we quantify the cost of naive diversification (i.e., the $1/n$ portfolio strategy).

In our calibration using U.S. industry portfolios data, we find that certainty equivalent costs of these three suboptimal strategies all increase with effective risk tolerance and are all convex functions of the non-normality ratio. These costs are important for investors who target large non-normality ratios whether positive or negative. The cost for ignoring returns non-normality is always higher than the cost for ignoring non-normality concerns, whereas the cost for naive diversification is more important that the cost for ignoring returns non-normality and increases with investor's effective risk tolerance as long as the non-normality ratio is sufficiently low. Otherwise, it is the opposite. Subsequently, we illustrate the performance of our optimal portfolio in a dynamic context. Concretely, we assume an investor who dynamically optimizes her portfolio to target the expected return and degree of non-normality of two trivial value-weighted rolling portfolio strategies. We find that the optimal approach enables a considerable reduction in the investor's risk exposure, and an increase to Sharpe ratio and certainty equivalent. On the other side, an investor willing to bear the same level of risk and asymmetry as observed on the value-weighted portfolio, and who considers investing one dollar in the optimally managed portfolio in January 1970, would have generated around one thousand dollars by November 2020, which represents five times the amount that would have been generated by the suboptimal strategy over the same period.

We finally derive the asset pricing implications of our model and find that expected returns are characterized by a linear two-beta model in the cross-section: the beta on the market portfolio

[1] In a mean-variance framework, Binsbergen et al. (2008) show that misalignment of objectives between a CFO and his asset managers can lead to large utility costs for the CFO.

(i.e., the standard CAPM beta) and the beta on the orthogonal portfolio. This second beta adds to the CAPM beta a component reflecting the asset's non-normality. In our calibration using U.S. industry portfolios data, we find that contrary to the prediction of the CAPM model, the CAPM beta explains at most 12% of the variation in asset risk premia while the second beta explains more than 60% of that variation. Overall, our results suggest that in an economy where asset returns are non-normally distributed and where investors have non-normality concerns when making investment decisions, the degree of asset non-normality is key to explain differences in expected returns across assets. These results emphasize the importance for asset pricing and capital budgeting of accounting for non-normality concerns in portfolio allocation.

## 2 The Multivariate Normal-Exponential Model of Asset Returns

The univariate normal-exponential distribution was first introduced by Aigner et al. (1977) to characterize the disturbance term in frontier production function models. A multivariate version of the model is presented by Adcock and Shutes (2012) with some properties that are relevant for portfolio choice and asset pricing in the presence of asymmetries in the key variables. Recent important portfolio choice and asset pricing studies featuring the multivariate normal-exponential model can be found in Dahlquist et al. (2016) and Schreindorfer (2019), respectively. Specifically, if the random vector $r = (r_1, r_2, \ldots, r_n)^\top$ of returns on $n$ risky assets follow an IID multivariate normal-exponential distribution, then they are described by the model:

$$r_i = \mu_i + \sigma_i \left[ \delta_i \left( e_0 - 1 \right) + \sqrt{1 - \delta_i^2} \varepsilon_i \right], \quad i = 1, 2, \ldots, n, \tag{1}$$

where the scalar random variable $e_0$ is a common shock across all assets and has an exponential distribution with a rate parameter equal to one[2] and the random vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^\top$ contains asset-specific shocks; it is independent of $e_0$ and has a standardized multivariate normal

[2] Adcock and Shutes (2012) show that the normal-exponential model is a certain limiting case of the extended skew-normal distribution and that the two models lead to very similar results in empirical applications.

distribution with correlation matrix $\Psi$.

By definition, the parameter $\mu_i$ is the mean of $r_i$, and the parameter $\sigma_i > 0$ is the volatility of $r_i$. Likewise, the parameter $\delta_i$, belonging to the interval $(-1, 1)$, determines the sensitivity of the asset return to the exponentially distributed common shock $e_0$. The exponential distribution is suitable for characterizing the occurrence of extreme events such as large and infrequent losses and gains. The waiting time until the next event in a Poisson process has an exponential distribution. The Poisson process is often used to characterize the occurrence of jumps in continuous-time models (see, e.g., Merton; 1976; Bates; 1996; and Broadie et al.; 2007). From equation (1), it follows that assets with large negative sensitivities to $e_0$ are subject to large but infrequent negative returns, whereas assets with large positive sensitivities are subject to large but infrequent positive returns. Model (1) assumes that the occurrence of such extreme movements is simultaneous across assets, so it may be interpretable as a systemic event. In this sense, current discrete-time return dynamics share the properties of the continuous-time dynamics considered by Das and Uppal (2004).

Parameters $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^\top$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)^\top$, $\Psi$, and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^\top$ together describe the return generating model. If $\delta = 0$, then $r$ follows a multivariate normal distribution with mean $\mu$, standard deviation vector $\sigma$, and correlation matrix $\Psi$. Hence, this setup conveniently nests the case when asset returns are jointly normal. In the extended model, $n$ extra parameters in $\delta$ are needed compared with the multivariate normal distribution; these additional parameters describe the asymmetry, or more generally, the non-normality in returns. Clearly, non-normality in our setting is generated by a single factor, $e_0$, but upon which different securities have different loadings. In total, the IID Model (1) has $n\,(n+5)\,/2$ parameters.

The mean, variance, skewness, and kurtosis of the return of asset $i$ are given by

$$\mathbb{E}\left[r_i\right] = \mu_i, \qquad \mathrm{var}\left[r_i\right] = \sigma_i^2, \qquad \mathrm{skew}\left[r_i\right] = 2\delta_i^3, \qquad \mathrm{kurt}\left[r_i\right] = 6\delta_i^4 + 3. \tag{2}$$

Likewise, the correlation, coskewness, and cokurtosis of the returns of asset $i$ and asset $j$ are

$$\begin{aligned} \operatorname{corr}\left(r_i, r_j\right) &= \delta_i \delta_j + \psi_{ij} \sqrt{1-\delta_i^2} \sqrt{1-\delta_j^2}, \qquad \operatorname{coskew}\left(r_i, r_j\right) = 2\delta_i \delta_j^2, \\ \operatorname{cokurt}\left(r_i, r_j\right) &= 6\delta_i \delta_j^3 + 3\left(\delta_i \delta_j + \psi_{ij} \sqrt{1-\delta_i^2} \sqrt{1-\delta_j^2}\right). \end{aligned} \tag{3}$$

The formulas in equations (2) and (3) show how vector $\delta$ characterizes the non-normality of returns, as it leads to non-zero skewness, coskewness, and excess kurtosis. Because the elements of $\delta$ all belong to the interval $(-1, 1)$, it appears that the multivariate normal-exponential distribution is well-suited when absolute values of skewness and coskewness are less than 2, and when kurtosis and cokurtosis values are less than 9. The parameters of the distribution can be estimated by the generalized method of moments (GMM) using the moments given in equations (2) and (3). They can alternatively be estimated by maximum likelihood. We restrict our attention in this paper to the GMM estimator because it enables investors to target some specific moments of asset returns they might be interested in. We later analyze its portfolio choice and asset pricing implications when the asset menu is composed of the U.S. industry portfolios and the risk-free rate.

Let's introduce the following parameters:

$$\begin{aligned} a &= \left(\sigma_1 \delta_1, \sigma_2 \delta_2, \ldots, \sigma_n \delta_n\right)^\top \\ D &= \operatorname{diag}\left(\sigma_1 \sqrt{1-\delta_1^2}, \sigma_2 \sqrt{1-\delta_2^2}, \ldots, \sigma_n \sqrt{1-\delta_n^2}\right) \quad \text{and} \quad \Omega = D \Psi D, \end{aligned} \tag{4}$$

where $\operatorname{diag}\left(g_1, g_2, \ldots, g_n\right)$ denotes a diagonal matrix with listed diagonal elements. One may write

$$r = \mu + a\left(e_0 - 1\right) + D\varepsilon. \tag{5}$$

It follows that conditional on $e_0$, the vector $r$ has a multivariate normal distribution with mean vector $\mu - a + a e_0$ and covariance matrix $\Omega$. Equation (5) is a new parametrization of the model where the vector of parameters is $\theta = \left(\mu^\top, \operatorname{vech}(\Omega)^\top, a^\top\right)^\top$. However, we can easily back out the original parameters from the estimates of $\mu$, $\Omega$ and $a$. Let $\omega$ be diagonal vector of $\Omega$. We have

$a_i = \sigma_i \delta_i$ and $\omega_i = \sigma_i^2 \left(1 - \delta_i^2\right) = \sigma_i^2 - a_i^2$, from which we obtain $\sigma_i = \sqrt{\omega_i + a_i^2}$. Once we have $\sigma_i$, we can compute $\delta_i$ as $\delta_i = a_i / \sigma_i$. Now that we have the vectors $\sigma$ and $\delta$, we easily obtain the matrix $D$ and compute the matrix $\Psi$ as $\Psi = D^{-1} \Omega D^{-1}$.

## 3 Generalized Method of Moments

We impose both symmetry and positive definiteness on $\Omega$ by writing $\Omega = XX^\top$, where $X$ is a lower triangular square matrix, thus having the same number of free parameters as $\Omega$. We can then reparameterize the vector $\theta$ by substituting out $\Omega$ by $X$. Our GMM estimator of the reparameterized vector of parameter $\theta = \left(\mu^\top, \mathrm{vech}\left(X\right)^\top, a^\top\right)^\top$ is based on matching the means, variances and covariances, skewness and coskewness, and kurtoses and cokurtoses of asset returns. In matrix form, we show that the moments in equations (2) and (3) may be written as

$$
\begin{aligned}
&\mathbb{E}\left[r\right] = \mu \\
&\mathbb{E}\left[\left(r - \mu\right)\left(r - \mu\right)^\top\right] = aa^\top + \Omega \\
&\mathbb{E}\left[\left(r - \mu\right)\left(\left(r - \mu\right) \odot \left(r - \mu\right)\right)^\top\right] = 2a\left(a \odot a\right)^\top \\
&\mathbb{E}\left[\left(r - \mu\right)\left(\left(r - \mu\right) \odot \left(r - \mu\right) \odot \left(r - \mu\right)\right)^\top\right] = 6a\left(a \odot a \odot a\right)^\top + 3\left(aa^\top + \Omega\right)\left(\left(aa^\top + \Omega\right) \odot I_n\right),
\end{aligned}
\tag{6}
$$

where $\odot$ denotes the Hadamard product, i.e., the element-by-element matrix multiplication $I_n$ is the $n \times n$ identity matrix. In total, equation (6) has $n\left(5n + 3\right)/2$ distinct moments that are used to estimate the $n\left(n + 5\right)/2$ parameters. Therefore, the number of moments exceeds the number of parameters by $n\left(2n - 1\right)$, meaning that our GMM estimation procedure is over-identified.

The moments in equation (6) are equivalent to $\mathbb{E}\left[g\left(r\right)\right] = 0$, where $g\left(r\right)$ is the following vector:

$$
g\left(r; \theta\right) = \begin{pmatrix}
r - \mu \\
\mathrm{vech}\left(\left(r - \mu\right)\left(r - \mu\right)^\top - \Sigma\right) \\
\mathrm{vec}\left(\left(r - \mu\right)\left(\left(r - \mu\right) \odot \left(r - \mu\right)\right)^\top - 2a\left(a \odot a\right)^\top\right) \\
\mathrm{vec}\left(\left(r - \mu\right)\left(\left(r - \mu\right) \odot \left(r - \mu\right) \odot \left(r - \mu\right)\right)^\top - 6a\left(a \odot a \odot a\right)^\top - 3\Sigma\Delta\right)
\end{pmatrix}, \tag{7}
$$

where $\Sigma = aa^\top + XX^\top$ is the covariance matrix of asset returns and $\Delta = \Sigma \odot I_n$ is the diagonal matrix with same diagonal elements as $\Sigma$, $\text{vec}(\cdot)$ is the vectorization of a matrix, i.e., a linear transformation that converts the matrix into a column vector, and $\text{vech}(\cdot)$ is the half-vectorization of a square matrix, obtained by vectorizing only the lower triangular part of the matrix. For a square matrix $M$, $\text{vech}(M)$ is obtained from $\text{vec}(M)$ by eliminating all supradiagonal elements of $M$.

Given an independent and identically distributed multivariate normal-exponential return series $\{r_t\}_{t=1}^T$, the GMM estimate of the vector of parameters $\theta = \left(\mu^\top, \text{vech}(X)^\top, a^\top\right)^\top$ is the solution to the following minimization problem:

$$\min_\theta T\overline{g(r;\theta)}^\top W(\theta)\,\overline{g(r;\theta)} \tag{8}$$

where $W(\theta)$ is the weighting matrix and it is understood that for a given function $h(x)$ we have

$$\overline{h(r)} = \frac{1}{T}\sum_{t=1}^T h(r_t).$$

Let $S(\theta)$ be the asymptotic variance-covariance matrix of $\overline{g(r;\theta)}$, i.e., the variance-covariance matrix of the limiting distribution of $\sqrt{T}\overline{g(r;\theta)}$. We denote $S(\theta) = \text{avar}\left(\overline{g(r;\theta)}\right)$. In our estimation, we use the weighting matrix $W(\theta) = \left(\hat{S}(\theta) \odot I_N\right)^{-1} L$, where $\hat{S}(\theta) = \widehat{\text{avar}}\left(\overline{g(r;\theta)}\right)$ is the Newey and West (1987) estimate of $S(\theta)$, $I_N$ is the identity matrix of size $N$, $L = \text{diag}(l_1, l_2, \ldots, l_N)$ is an $N \times N$ diagonal matrix with logical elements, and where $N = n(5n+3)/2$ is the size of $g(r;\theta)$. With this weighting matrix, equation (8) is equivalent to

$$\min_\theta \sum_{i=1}^N l_i \frac{\overline{g_i(r;\theta)}^2}{\widehat{\text{avar}}\left(\overline{g_i(r;\theta)}\right)/T}, \tag{9}$$

where $l_i = 1$ for moments that are used for parameter estimation and $l_i = 0$ otherwise. Thus, our assumed weighting matrix assigns more weight to moments with smaller variance. Notice that our weighting matrix is also endogenous, i.e., the vector of parameters and the spectral matrix are estimated simultaneously (see discussion in Cochrane; 2009, Section 11.7). Therefore, our GMM procedure is similar to the continuous-updating GMM approach proposed by Hansen et al. (1996).

We consider five different specifications of the GMM procedure. In **Sp1**, $l_i = 1$ only for means, variances and covariances, and skewness, a total of $N_1 = n\,(n+5)\,/2$ moments. Thus, **Sp1** has as many moment conditions as the number of parameters and is, therefore, an identified GMM. In **Sp2**, in addition to moments considered in **Sp1**, $l_i = 1$ only for coskewness, a total of $N_2 = 3n\,(n+1)\,/2$ moments. In **Sp3**, in addition to moments considered in **Sp1**, $l_i = 1$ only for kurtosis, a total of $N_3 = n\,(n+7)\,/2$ moments. In **Sp4**, in addition to moments considered in **Sp2**, $l_i = 1$ only for kurtosis, a total of $N_4 = n\,(3n+5)\,/2$ moments. Finally, in **Sp5**, $l_i = 1$ for all moments, a total of $N_5 = N = n\,(5n+3)\,/2$ moments.

Given the estimates of model parameters, say the vector $\hat{\theta}$, their asymptotic covariance matrix $\hat{V}$ is estimated as usual, i.e., as follows:

$$\hat{V} = \frac{1}{T}\left(\hat{G}^{\top}\hat{W}\hat{G}\right)^{-1}\hat{G}^{\top}\left(\hat{W}\hat{S}\hat{W}\right)\hat{G}\left(\hat{G}^{\top}\hat{W}\hat{G}\right)^{-1} \tag{10}$$

where $\hat{G} = \left.\frac{\partial \overline{g\,(r;\theta)}}{\partial \theta^{\top}}\right|_{\theta=\hat{\theta}}$, $\hat{W} = W\left(\hat{\theta}\right)$ and $\hat{S} = \hat{S}\left(\hat{\theta}\right)$.

# 4 Data and Estimation Results

We empirically explore our proposed estimation schemes and analyze the properties of the estimators using stock data on U.S. industry portfolios downloaded from the Kenneth French's data library.[3] The original data is monthly; the 49 industry portfolios database spans the period from July 1926 to November 2020, which is a total of $T = 1,133$ months. With $n = 49$, there would be 1,323 parameters, which is larger than the number of periods. To keep the number of parameters sufficiently low with respect to the number of periods, we select returns that simultaneously have a

[3] `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

positive skewness smaller than 2 and a positive excess kurtosis smaller than 6. In particular, these conditions satisfy the third- and fourth-moment bounds of the normal-exponential distribution as discussed in Section 2. We obtain 11 industry portfolios satisfying this criterion. These industries are Food Products (**Fd**), Apparel (**Cl**), Construction Materials (**Bm**), Non-Metallic and Industrial Metal Mining (**Mn**), Petroleum and Natural Gas (**Oi**), Computers (**Hw**), Electronic Equipment (**Ch**), Shipping Containers (**Bx**), Retail (**Rt**), Banking (**Bk**) and Other (**Ot**).

With $n = 11$, there are 88 parameters, i.e., 88 moment conditions in the case of an identified GMM such as in **Sp1**. There is long-standing debate in the econometrics literature as to what number of moment conditions would be reasonable for given sample size in the GMM estimation procedure, i.e., for the usual limiting distribution of the GMM estimator of a fixed number of parameters to remain valid. Koenker and Machado (1999) establish that for a class of linear models with general heteroscedasticity, $N_T = o\left(T^{1/3}\right)$ is a sufficient condition. They argue that it is difficult to imagine that more complicated, nonlinear GMM models, like the one characterizing our multivariate normal-exponential distribution, could sustain a faster rate of growth of the number of moment conditions in terms of the sample size than the one derived for the linear case. However, the Koenker and Machado (1999) result is not a necessary condition for valid GMM inference in a finite sample. Therefore, the number of moment conditions in our GMM specifications (88 for **Sp1**, 198 for **Sp2**, 99 for **Sp3**, 209 for **Sp4**, and 319 for **Sp5**) may not meet this sufficient condition.[4]

Summary statistics of asset returns and an overview of the ability of the multivariate normal-exponential model GMM estimates to match sample skewness and kurtosis are presented in Table 1. In particular, as shown in Panel B of the table, identified GMM estimates from **Sp1** match perfectly the sample skewness. Because values of the sample skewness are small relative to the maximum value of 2 allowed by the model, the implied values of the parameter vector $\delta$ are small, implying a small asset kurtosis. In **Sp3** and **Sp5**, we ask the GMM estimates to match as many sample

[4]Therefore, a next step would be to conduct simulation experiments to analyze the small sample properties of our estimators. We do not pursue this in the current article as our GMM estimation procedure appears to run slowly and this exercise would be more valuable in a purely econometrics work which is beyond the scope of the paper.

third-order moments ((co)-skewness) as fourth-order moments ((co)-kurtosis) jointly; therefore, the implied values of asset kurtosis are higher compared with other specifications where there are fewer fourth-order moments as shown in Panel C. In general, because all higher-order (co)-moments depend on the single parameter vector $\delta$, it is obvious that their sample counterparts cannot all be matched perfectly by the GMM estimates and it would be at the discretion of the econometrician and the model user to decide which key moments should be taken into account depending on the model application. Sample kurtosis would be better matched at the expense of sample skewness, for example, in specifications where there would be more fourth-order moments than third-order moments, resulting in higher implied values of the parameter vector $\delta$.

We now discuss the statistical significance of the estimated parameters from our proposed GMM procedure. The estimates of the parameter vectors $\mu$ and $a$ with their standard errors are presented in Panel A and Panel B of Table 2, respectively, for all five GMM specifications. Panel C displays the values of the parameter vectors $\sigma$ and $\delta$ implied by the GMM estimates. Panel A shows that estimates of asset means are highly significant. Comparing these mean estimates to the sample means reported in Panel A of Table 1, we observe that the identified GMM mean estimates of **Sp1** match perfectly the sample means as expected and that GMM mean estimates are relatively stable across **Sp1** to **Sp4**. Large discrepancies between GMM mean estimates and sample means occur when adding cokurtosis to moment conditions. The same observation holds when we compare model-implied asset volatilities of Panel C to the sample volatilities reported in Panel A of Table 1. GMM estimates of the degree of non-normality $a$ as shown in Panel B of Table 2 are less stable across specifications because all higher-order moments depend on $a$; otherwise, the parameter vector $a$ is not identifiable and its identification would depend on which higher-order moments are included in the moment conditions. In terms of inference, we observe that adding kurtosis to the moment conditions generally improves the statistical significance of the GMM estimates of $a$. Likewise, the implied values of the parameter vector $\delta$ as shown in Panel C of Table 2 are the largest for **Sp3** across all GMM specifications.

The identified GMM estimate of the triangular matrix $X$ from **Sp1** is shown in Panel D of Table 2 while Table 3 displays GMM estimates of $X$ for the remaining specifications. It also holds that statistical significance is improved overall when kurtosis is added to the moment conditions. For example, it can easily be picked when comparing Panel D of Table 2 with Panel B of Table 3 that standard errors of the GMM estimates of $X$ from **Sp3** are much lower compared with standard errors of the GMM estimates of $X$ from **Sp1**. Several elements of the matrix $X$ that are not statistically significant under **Sp1** are highly significant under **Sp3** after kurtosis has been added to the moment conditions. In the remainder of the article, we examine an asset allocation problem with its asset pricing implications when asset returns are generated from a multivariate normal-exponential model. We numerically illustrate our findings using the 11 industry portfolios retained in the current section and rely on GMM estimates from **Sp3**.

# 5 Portfolio Choice Solution: Single Investor's Setting

In a single-period economy, we consider the problem of an investor who faces the asset menu of Section 2 and wants to optimally choose a portfolio strategy. It is straightforward to show that, for asset net returns characterized by the return generating Model (1), the simple net return of a portfolio strategy $w$, given by $r_w = r_f + w^\top \left(r - \mathbf{1} r_f\right)$, is characterized by the normal-exponential model:

$$r_w = \mu_w + \left(\sigma_w \delta_w\right)\left(e_0 - 1\right) + \left(\sigma_w \sqrt{1-\delta_w^2}\right)\varepsilon_w, \tag{11}$$

with

$$\mu_w = r_f + w^\top \left(\mu - \mathbf{1} r_f\right), \qquad \sigma_w^2 = w^\top \Sigma w, \qquad \delta_w = \frac{w^\top \left(\sigma \odot \delta\right)}{\sigma_w}. \tag{12}$$

More generally, the multivariate normal-exponential distribution is closed under linear transformations (see also Adcock and Shutes; 2012). Exploiting this property, let's consider the linear span of the $n+1$ asset returns, i.e, $\mathcal{S} = \mathrm{Span}\left(r_f, r_1, r_2, \ldots, r_n\right)$. In the assumed economy, returns on portfolio strategies are a subset of $\mathcal{S}$. We endow the return space $\mathcal{S}$ with the non-normality operator nnorm $[\cdot]$ such that nnorm $\left[r_p\right] = \sigma_p \delta_p$ for any $r_p \in \mathcal{S}$, where $\sigma_p$ and $\delta_p$ are, respectively, the volatility and the non-normality coefficient of $r_p$. We argue, and it can easily be proved that nnorm $[\cdot]$ is a linear operator.

We now consider the problem of choosing the portfolio strategy that minimizes the portfolio variance var $[r_w]$ subject to the following constraints on portfolio expected return and degree of non-normality: $\mathbb{E}[r_w] \geq \mu_w$ and nnorm $[r_w] = \sigma_w \delta_w$. Formally, the problem is stated as follows:

$$\min_w \frac{1}{2} w^\top \Sigma w \qquad \text{subject to} \qquad w^\top \left(\mu - \mathbf{1} r_f\right) \geq \mu_w - r_f, \qquad w^\top (\sigma \odot \delta) = \sigma_w \delta_w, \tag{13}$$

where $\mu_w$ and $\sigma_w \delta_w$ are given. We show that the solution to the optimal portfolio choice problem (13) may be written:

$$w = \lambda_1 \Sigma^{-1} \left(\mu - \mathbf{1} r_f\right) + \lambda_2 \Sigma^{-1} (\sigma \odot \delta), \tag{14}$$

where $\lambda_1$ and $\lambda_2$ are the Karush–Kuhn–Tucker multipliers associated with the mean and the non-normality constraints of problem (13), respectively. Values of the Karush–Kuhn–Tucker multipliers $\lambda_1$ and $\lambda_2$ depend on whether the inequality constraint is or is not binding when evaluated at the optimal portfolio $w$ of equation (14).

We prove in the appendix that the Karush–Kuhn–Tucker multipliers $\lambda_1$ and $\lambda_2$ are given by:

$$\lambda_1 = i_w \lambda_{11} \quad \text{and} \quad \lambda_2 = i_w \lambda_{21} + (1 - i_w) \lambda_{22}, \tag{15}$$

where

$$\begin{cases} \lambda_{11} & = \dfrac{A}{AC - B^2} \left(\mu_w - r_f\right) - \dfrac{B}{AC - B^2} \sigma_w \delta_w \\[2ex] \lambda_{21} & = -\dfrac{B}{AC - B^2} \left(\mu_w - r_f\right) + \dfrac{C}{AC - B^2} \sigma_w \delta_w \end{cases} \quad \text{and} \quad \lambda_{22} = \frac{1}{A} \sigma_w \delta_w \tag{16}$$

and where

$$
\begin{aligned}
i_w &= \mathbb{I}\left(\mu_w \geq r_f + \frac{B}{A}\sigma_w\delta_w\right), \qquad & A &= (\sigma \odot \delta)^\top \Sigma^{-1} (\sigma \odot \delta), \\
B &= \left(\mu - \mathbf{1}r_f\right)^\top \Sigma^{-1} (\sigma \odot \delta), \qquad & C &= \left(\mu - \mathbf{1}r_f\right)^\top \Sigma^{-1} \left(\mu - \mathbf{1}r_f\right).
\end{aligned} \tag{17}
$$

Notice that $\mathbb{I}(\cdot)$ denotes the indicator function. It is obvious that $A \geq 0$ and $C \geq 0$. It is also obvious from the Cauchy-Schwarz inequality that we have $AC - B^2 \geq 0$. In applications, there is in general at least one asset whose return distribution deviates from normality, i.e., $\sigma_i\delta_i \neq 0$, ensuring that $A > 0$. Also, there is at least one asset whose expected return is different from the risk-free rate, i.e., such that $\mu_i - r_f \neq 0$, ensuring that $C > 0$. Likewise, the vector of asset risk premia will in general not be linear in the vector of asset non-normality measures, i.e., we cannot find a constant scalar coefficient $d$ such that $\mu - \mathbf{1}r_f = (\sigma \odot \delta)\, d$, ensuring that $AC - B^2 > 0$.

### 5.1 Funds' separation and optimal non-participation

Equation (14) may also be expressed as:

$$
w = \alpha_1 w^{\mathbf{MV}} + \alpha_2 w^{\mathbf{AV}} \tag{18}
$$

where

$$
\begin{aligned}
w^{\mathbf{MV}} = \frac{\Sigma^{-1}\left(\mu - \mathbf{1}r_f\right)}{\mathbf{1}^\top\Sigma^{-1}\left(\mu - \mathbf{1}r_f\right)} \quad &\text{and} \quad w^{\mathbf{AV}} = \frac{\Sigma^{-1}(\sigma \odot \delta)}{\mathbf{1}^\top\Sigma^{-1}(\sigma \odot \delta)} \\
\alpha_1 = \mathbf{1}^\top\Sigma^{-1}\left(\mu - \mathbf{1}r_f\right)\lambda_1 \quad &\text{and} \quad \alpha_2 = \mathbf{1}^\top\Sigma^{-1}(\sigma \odot \delta)\,\lambda_2.
\end{aligned} \tag{19}
$$

Notice that $w^{\mathbf{MV}}$ and $w^{\mathbf{AV}}$ are two mutual funds fully invested in risky securities and $\alpha_1$ and $\alpha_2$ are optimal positions in these mutual funds, respectively. The investment in the risk-free security is, therefore, $1 - \mathbf{1}^\top w$ or equivalently $1 - \alpha_1 - \alpha_2$. The first risky fund, $w^{\mathbf{MV}}$, is the solution to the mean-variance optimal portfolio problem, i.e., a problem like (13) where only the mean constraint is active. Likewise, the second risky fund, $w^{\mathbf{AV}}$, is the solution to a non-normality-variance optimal portfolio problem similar to the mean-variance one, i.e., a problem like (13) where only the non-normality constraint is active.

Using the asset menu composed of the final 11 industry portfolios discussed earlier and relying on parameter values corresponding to GMM estimates from **Sp3**, Table 4 displays the parameters used to calibrate the asset returns distribution together with some key statistics of the two mutual funds as well as their weights in each of the individual assets. Observe from the last two rows of Table 4 that the **MV** fund has a higher Sharpe ratio (equal to 0.18) than any of the individual assets (their maximum is 0.14, for **Fd**). This is achieved by large long positions on assets such as **Fd**, **Bx**, **Rt**, **Hw**, and **Bk**, which have the highest Sharpe ratios among available assets. The **MV** fund weights of these five industry assets are 63%, 20%, 15%, 24%, and 18%, respectively, as shown in the last two columns of the table. Likewise, the **AV** fund has a higher positive non-normality coefficient (equal to 0.81) than any of the individual assets (their maximum is 0.67, for **Ch**). This is achieved by large long positions on assets such as **Ch**, **Bm**, **Cl**, and **Oi**, which have the highest non-normality coefficients among available assets. The **AV** fund weights of these four industry assets are 52%, 48%, 61%, and 38%, respectively. An investor who targets a positive degree of non-normality would likely take a sufficiently long position in the **AV** fund, whereas an investor who targets a negative degree of non-normality would likely take a sufficiently short position in the **AV** fund. Interestingly, characteristics of the two funds are not driven by the same industry assets. Actually, not a single industry asset among the leading long positions in any of the two funds appears to be a leading long position in the other. The **AV** fund weights of the five leading **MV** fund long positions are 1%, 0%, -57%, -7%, and -16%, respectively. We see this as a portfolio diversification potential.

Equation (18) suggests that investment in risky assets is made via the two risky funds; thus, overall investment satisfies a three-fund separation theorem. In our $(n+1)$-asset economy where the $n$ risky asset returns are multivariate normal-exponential, investors who seek to minimize portfolio variance subject to given level of expected return and degree of return non-normality would all invest a proportion $\alpha_1$ of their wealth in the **MV** fund, a proportion $\alpha_2$ in the **AV** fund, and the remaining proportion $1 - \alpha_1 - \alpha_2$ in the safe asset. This finding is equivalent to Dahlquist et al. (2016) and similar to Simaan (1993).

We finally would like to characterize optimal non-participation in risky security markets such as in Ang et al. (2005) in their portfolio choice framework featuring disappointment aversion. In our setting, investors differ through their targets, i.e., through the minimum expected return and degree of non-normality they want to achieve with their optimal portfolio. From our optimal three-fund separation strategy, it is obvious that non-participation is equivalent to $\lambda_1 = \lambda_2 = 0$, i.e., the investor takes zero position in each of the two mutual funds. It is straightforward from equations (15) to (17) that $\lambda_1 = \lambda_2 = 0$ is equivalent to $\sigma_w \delta_w = 0$ and $\mu_w - r_f \leq 0$. Thus, optimal non-participation in risky security markets in our setting is equivalent to seeking an optimal portfolio that is normally distributed and has a non-positive risk premium. This shows that earlier analytical results about optimal non-participation in the case of a single security as in the disappointment aversion preferences setting of Ang et al. (2005) also extend to the setup with multiple risky assets.[5]

### 5.2 Endogenous preference parameters

Dahlquist et al. (2016) consider an investor with generalized disappointment aversion (GDA) preferences and who maximizes the certainty equivalent of her portfolio when available asset returns are generated by a multivariate normal-exponential model. GDA preferences are characterized by three key parameters: the curvature of the utility function ($\gamma$), the penalty attributed to disappointing events ($\ell$), and the percentage of the certain equivalent below which the investor is disappointed ($\kappa$). Comparing our optimal portfolio solution (14) to equations (17) and (18) of Dahlquist et al. (2016), their observational equivalence gives rise to the interpretation of the multipliers $\lambda_1$ and $\lambda_2$ as endogenous preference parameters. Therefore, we can interpret $\lambda_1$ as the investor's risk tolerance coefficient and $\lambda_2$ as the investor's non-normality concern coefficient.

[5] Dahlquist et al. (2016) also illustrate optimal non-participation with multiple securities in the disappointment aversion preferences framework without providing an analytical characterization of it.

In our setting, risk tolerance and non-normality concern coefficients are endogenous because they depend on the asset menu through the coefficients $A$, $B$, and $C$, as well as the minimum level of expected return $\mu_w$ and the degree of non-normality $\sigma_w \delta_w$ that the investor wants to achieve. Thus, when setting the targets (minimum expected return and degree of non-normality) characterizing her optimal portfolio, the investor implicitly reveals her underlying preference parameters for a given asset menu. We argue that these measures of reward and risk that the investor ultimately seeks to achieve in the long run via the chosen portfolio seem more realistic to think of when it comes to characterizing and representing people's attitude toward risk. Likewise, these measures are easier to imagine, assess and understand compared with more abstract preference parameter values such as the curvature of the utility function, the penalty attributed to disappointing events, and the percentage of the certain equivalent below which the investor is disappointed.

We have that

$$\begin{aligned} \frac{\partial \lambda_1}{\partial \mu_w} &= \frac{i_w A}{AC - B^2} \quad \text{and} \quad \frac{\partial \lambda_1}{\partial \left(\sigma_w \delta_w\right)} = -\frac{i_w B}{AC - B^2} \\ \frac{\partial \lambda_2}{\partial \mu_w} &= -\frac{i_w B}{AC - B^2} \quad \text{and} \quad \frac{\partial \lambda_2}{\partial \left(\sigma_w \delta_w\right)} = \frac{i_w C}{AC - B^2} + \frac{1 - i_w}{A}. \end{aligned} \tag{20}$$

Equation (20) determines how endogenous preference parameters vary with the investor's targets. In particular, it suggests that risk tolerance weakly increases with the level of expected return requested by the investor. Therefore, more risk-tolerant investors turn out to be those not willing to achieve a lower portfolio expected return. Likewise, the investor's non-normality concern strictly increases with the degree of non-normality that she would like to achieve.

All endogenous quantities depend on the investor's targets $\mu_w$ and $\sigma_w \delta_w$. By observing that $\sigma_w \delta_w = \frac{\delta_w}{S_w} \left(\mu_w - r_f\right)$, we can alternatively express them in terms of $\mu_w - r_f$ and $\frac{\delta_w}{S_w}$, where $S_w$ denotes the portfolio's Sharpe ratio. We therefore graphically represent all endogenous portfolio-related quantities for different values of $\frac{\delta_w}{S_w}$. We consider values of $\frac{\delta_w}{S_w}$ satisfying $\frac{\delta_w}{S_w} < \frac{A}{B}$, so that $i_w = \mathbb{I}\left(\mu_w - r_f \geq 0\right)$. We further refer to $\frac{\delta_i}{S_i}$ as the non-normality ratio. This ratio is given in Table 4 for each individual industry portfolio in our sample and ranges from 3.25 for **Fd** to 8.92 for **Ot**.

Likewise, the non-normality ratios of the **MV** and the **AV** funds are 2.48 and 8.40, respectively.

We start by plotting the endogenous preferences parameters in Panel A of Figure 1 for the effective risk tolerance coefficient and in Panel B for the effective non-normality concern coefficient. As expected, as the targeted risk premium increases everything else being equal, the more risk-tolerant the investor. Likewise, if two investors target the same level of portfolio risk premium, then the more conservative would be the investor with the positive non-normality ratio. For example, among investors who target the same portfolio risk premium of 0.5% per month, the effective risk tolerance coefficients are 0.3632 and 0.0921; i.e., the effective risk aversion coefficients are 2.7531 and 10.8531 for the two investors with targeted non-normality ratios of -5 and 5, respectively. Also observe that, the larger the magnitude of the non-normality ratio, the more extreme the investor's behavior (i.e., conservative for a positive non-normality ratio and, aggressive for a negative non-normality ratio). For example, among investors who target the same portfolio risk premium of 0.5% per month, the investor with the targeted non-normality ratio of -5 is more aggressive than the investor with the targeted non-normality ratio of -3 and for whom effective risk tolerance coefficient is 0.3090 (i.e., effective risk aversion coefficient is 3.2361). Likewise, the investor with the targeted non-normality ratio of 5 is more conservative than the investor with the targeted non-normality ratio of 3 and for whom effective risk tolerance coefficient is 0.1464 (i.e., effective risk aversion coefficient is 6.8326).

Non-participation is well illustrated in Panels A and B of Figure 1. It is clear from the left part of the graphs (i.e., for $\mu_w - r_f \leq 0$) that $\lambda_1 = 0$ for all non-normality ratio levels, and that $\lambda_2 = 0$ only if the non-normality ratio is equal to zero. When $\lambda_1 = 0$, the zero non-normality ratio that characterizes optimal non-participation is actually a knife-edge case in the model; as for investors with nonzero non-normality ratio, investing in the risk-free asset only is never optimal, i.e., $\lambda_2 \neq 0$ as illustrated in Panel B of Figure 1. From the right part (i.e., for $\mu_w - r_f > 0$) of the graphs of Panel B, we observe as expected from equation (20) that the effective non-normality concern coefficient strictly increases with the non-normality ratio. The effective non-normality concern coefficient is positive when the non-normality ratio is positive and negative otherwise.

Panels C and D of Figure 1 display the optimal portfolio weights in the **MV** fund and the **AV** fund, respectively. These weights are plotted against the effective risk tolerance coefficient for different values of the non-normality ratio. As expected from equation (19), all investors with the same effective risk tolerance $\lambda_1$ will invest the same fraction $\alpha_1$ of their respective wealths in the **MV** fund regardless of their targeted portfolio characteristics. For example, consider the following three investors. Investor 1 has targeted risk premium and non-normality ratio of 0.28% per month and -5, respectively. Investor 2 has targeted risk premium and non-normality ratio of 0.44% per month and 0, respectively. Investor 3 has targeted risk premium and non-normality ratio of 1.09% per month and 5, respectively. The effective risk tolerance coefficients of these three investors are 0.2005, 0.2005, and 0.2009, respectively; i.e., they are all about equal despite significant heterogeneity in their targeted portfolio characteristics. These three investors will invest about the same 67.50% of their wealth in the **MV** fund, but different fractions in the **AV** fund. Therefore, the concept of effective risk tolerance provides a convenient way to compare the effects of different preferences as represented by the investor's targeted portfolio characteristics in the presence of returns non-normality and investor's non-normality concerns.

Comparing the optimal choices of different investors (e.g., targeted normally distributed portfolio versus targeted non-normally distributed portfolio) who have the same effective risk tolerance isolates the effect of returns non-normality, as such investors would choose the same portfolios if returns were normally distributed. Regarding our three illustrative investors with same effective risk tolerance, Investor 1 has a preference for a negative degree of non-normality and, as such, she chooses to hold a short position in the **AV** fund, amounting to 48.49% of her wealth. Investor 2 has a preference for a normally distributed portfolio and as such, she chooses to hold a more moderate short position in the **AV** fund, amounting to 25.65% of her wealth. In contrast, Investor 3 has a preference for a positive degree of non-normality and as such, she chooses to hold a long position in the **AV** fund, amounting to 64.52% of her wealth. In the following, we measure the financial welfare generated by investor portfolio decisions and the cost for making suboptimal choices.

### 5.3 Certainty equivalent

We use the concept of the certainty equivalent to measure the financial welfare generated by investor portfolio decisions as typical in the portfolio choice literature. Interestingly, equation (14) is also equivalent to the solution of an equivalent portfolio choice problem where the investor, endowed with given risk tolerance coefficient $\lambda_1$ and non-normality concern coefficient $\lambda_2$ maximizes the following mean-variance-non-normality certainty equivalent (expressed in variance units):

$$\mathcal{R}_w \equiv \lambda_1 \mathbb{E}\left[r_w - r_f\right] - \frac{1}{2}\operatorname{var}\left[r_w\right] + \lambda_2 \operatorname{nnorm}\left[r_w\right]. \tag{21}$$

In expected return (or mean) units, the certainty equivalent would be $\mathcal{R}_w/\lambda_1$ for $\lambda_1 > 0$. In mean units, this certainty equivalent measures how much the investor would get on top of the risk-free rate to be indifferent about holding the optimal portfolio. We therefore can use the mean-variance-non-normality certainty equivalent (21) as a criterion for comparing different portfolio strategies faced by an investor with given preference parameters. The use of certainty equivalents for portfolio comparison is very common in the literature (see for example, Das and Uppal; 2004 and Dahlquist et al.; 2016 among others). In particular, we compute the certainty equivalent of the optimal portfolio strategy (14) and find that

$$\mathcal{R}_w = \frac{1}{2}\sigma_w^2. \tag{22}$$

Panel A of Figure 2 displays the certainty equivalent of the optimal portfolio in mean units. This financial welfare measure is plotted against the effective risk tolerance coefficient for different values of the non-normality ratio. These plots suggest that everything else being equal, the certainty equivalent of the optimal portfolio increases with the investor's risk tolerance. Likewise, the certainty equivalent of the optimal portfolio is a convex function of the non-normality ratio, with a minimum when the non-normality ratio is zero. This latter observation is confirmed in Panel A of

Figure 3, where the optimal certainty equivalent is alternatively plotted against the non-normality ratio for different levels of investor's effective risk tolerance. Let's consider again three illustrative investors, now with unitary risk tolerance (i.e., $\lambda_1 = 1$), and the same non-normality ratios as previously considered (i.e., -5 for Investor 1, 0 for Investor 2, and 5 for Investor 3), and let's multiply the certainty equivalent values by 1,000 so that they indicate the financial welfare for an investor with an initial wealth of \$1,000. Therefore, the optimal certainty equivalent values are \$14.62, \$10.99, and \$67.83 for Investor 1, Investor 2, and Investor 3, respectively. At equal effective risk tolerance, and with respect to the investor with zero non-normality ratio, the certainty equivalent is considerably higher for an investor with a positive non-normality ratio compared with the investor with a non-normality ratio that is opposite.

We now turn to the cost incurred by investors for making suboptimal choices. For any (suboptimal) portfolio $w'$ and given the endogenous preference parameters $\lambda_1$ and $\lambda_2$ of equation (16) from the same asset menu and targets, we can compute the associated certainty equivalent as follows: $\mathcal{R}_{w'} = \lambda_1 w'^\top \left(\mu - \mathbf{1} r_f\right) - \frac{1}{2} w'^\top \Sigma w' + \lambda_2 w'^\top \left(\sigma \odot \delta\right)$. The difference $\mathcal{R}_w - \mathcal{R}_{w'}$ shall be positive and represents the certainty equivalent cost of choosing the suboptimal allocation instead of the optimal allocation. We now discuss three special suboptimal choices for which welfare costs in mean units are plotted in the remaining panels of Figure 2.

**Ignoring returns non-normality** First, suppose the investor for whom endogenous preference parameters $\lambda_1$ and $\lambda_2$ of equation (16) are given, ignores the non-normality of asset returns, i.e., she chooses to hold the same portfolio as a mean-variance investor with the same effective risk tolerance. That is, the investor chooses to hold the portfolio $w' = \lambda_1 \Sigma^{-1} \left(\mu - \mathbf{1} r_f\right)$ instead of the optimal portfolio $w$. The former allocation is suboptimal and the difference $\mathcal{R}_w - \mathcal{R}_{w'}$ shall be positive and represents the certainty equivalent cost of ignoring returns non-normality. Similar to the optimal certainty equivalent, the certainty equivalent cost for ignoring returns non-normality is a convex function of the non-normality ratio. It achieves the minimum value of zero for some positive value of the non-normality ratio, closer to 3 in the current calibration, as can be inferred from Panel B of

Figure 2, and confirmed from Panel B of Figure 3. Out of a \$1,000 initial investment, these costs are \$16.44, \$4.61, and \$29.06 for Investor 1, Investor 2, and Investor 3, respectively.

**Ignoring non-normality concerns** Next, suppose the investor for whom endogenous preference parameters $\lambda_1$ and $\lambda_2$ of equation (16) are given, ignores non-normality concerns, i.e., she does not consider the non-normality constraint in her portfolio optimization problem, thus behaving like a mean-variance investor. Mean-variance optimization leads the investor to choose the portfolio $w' = \frac{\max\left(\mu_w - r_f, 0\right)}{C} \Sigma^{-1}\left(\mu - \mathbf{1} r_f\right)$ instead of the optimal portfolio $w$. The former allocation is suboptimal and the difference $\mathcal{R}_w - \mathcal{R}_{w'}$ shall be positive and represents the certainty equivalent cost of ignoring non-normality concerns. As shown in Panel C of Figure 2 as well as in Panel C of Figure 3, this certainty equivalent cost has the same pattern as the cost for ignoring returns non-normality, but is lower. Considering again our three illustrative investors with unitary effective risk tolerance coefficient, their certainty equivalent costs for ignoring non-normality concerns are \$11.58, \$3.25, and \$20.48 for Investor 1, Investor 2, and Investor 3, respectively.

**Naive diversification** Finally, suppose the investor whose endogenous preference parameters $\lambda_1$ and $\lambda_2$ of equation (16) are given, decides to go for a rough and, more or less, instinctive common-sense division of a portfolio, without bothering with sophisticated mathematical models, i.e., she follows the $1/n$ rule. In this case, the investor chooses to hold the portfolio $w' = \left(w^\top \mathbf{1}\right) \frac{1}{n} \mathbf{1}$ instead of the optimal portfolio $w$. The former allocation is suboptimal and the difference $\mathcal{R}_w - \mathcal{R}_{w'}$ shall be positive and represents the certainty equivalent cost of naive diversification. Likewise, as shown in Panel D of Figure 2, this cost increases with the investor's risk tolerance everything else being equal. Similarly, it is a convex function of the investor's non-normality ratio, achieving its minimum value at some positive value of the non-normality ratio, also closer to 3 in the current calibration, as confirmed in Panel D of Figure 3. In particular, considering again our three illustrative investors with unitary effective risk tolerance coefficients, their certainty equivalent costs for naive diversification are \$16.88, \$11.51 and \$13.61 for Investor 1, Investor 2, and Investor 3, respectively. In the current calibration, the certainty equivalent cost for naive diversification is more important that the certainty equivalent cost for ignoring returns non-normality and increases with the investor's effective risk tolerance as long as the non-normality ratio is sufficiently low. Otherwise, it is the opposite. The two panels of Figure 4 confirm this latter observation.

### 5.4 Mean-variance-non-normality frontier and efficient frontier

We now aim at characterizing the set of optimal portfolios when varying investors' targeted portfolio characteristics, called the "frontier" as usually understood. Starting from the optimal portfolio strategy $w = \lambda_1 \Sigma^{-1} \left(\mu - \mathbf{1} r_f\right) + \lambda_2 \Sigma^{-1} (\sigma \odot \delta)$ in equation (14), we compute its variance $\sigma_w^2 = w^\top \Sigma w$ and obtain $\sigma_w^2 = A\lambda_2^2 + 2B\lambda_1\lambda_2 + C\lambda_1^2$. It follows that the optimal portfolio variance is a quadratic function of the multipliers. By "completing the square" used in factoring quadratic polynomials, we have $\sigma_w^2 = \frac{(A\lambda_2 + B\lambda_1)^2}{A} + \frac{\left(AC - B^2\right)\lambda_1^2}{A}$. Substituting out $\lambda_1$ and $\lambda_2$ by their expressions in terms of $\mu_w - r_f$ and $\sigma_w \delta_w$ as given in equation (16), we obtain $\sigma_w^2 = \frac{(\sigma_w \delta_w)^2}{A} + \frac{\left(A\left(\mu_w - r_f\right) - B\sigma_w\delta_w\right)^2}{A\left(AC - B^2\right)}$. Dividing both sides of this latter equation by $\sigma_w^2$ we obtain

$$\frac{\delta_w^2}{A} + \frac{(AS_w - B\delta_w)^2}{A\left(AC - B^2\right)} = 1, \tag{23}$$

where $S_w$ is the portfolio's Sharpe ratio and $\delta_w$ is the portfolio's non-normality parameter. Equation (23) characterizes the mean-variance-non-normality frontier. It shows that the mean-variance-non-normality frontier is an ellipse in the (non-normality, Sharpe ratio) space, or $(\delta, S)$-space.

The two optimal mutual funds **MV** and **AV** can easily be materialized on the mean-variance-non-normality frontier. Notice that if investment in risky assets is fully made in the **MV** fund, this requires $\lambda_2 = 0$ or, equivalently, $S_w = \frac{C}{B}\delta_w$. The **MV** fund is thus at the intersection between the mean-variance-non-normality frontier and the line with slope $C/B$ that goes through the origin in the $(\delta, S)$-space. Similarly, if investment in risky assets is fully made in the **AV** fund, then this requires $\lambda_1 = 0$ or, equivalently, $S_w = \frac{B}{A}\delta_w$. The **AV** fund is thus at the intersection between the frontier and the line with slope $B/A$ that goes through the origin in the $(\delta, S)$-space.

The efficient frontier is the portion of the mean-variance-non-normality frontier comprising portfolios chosen by investors with nonnegative risk tolerance, i.e., portfolios such that $\lambda_1 \geq 0$ or, equivalently, $S_w \geq \frac{B}{A}\delta_w$. Thus, the efficient frontier is the part the ellipse that is above the line with slope $B/A$ that goes through the origin in the $(\delta, S)$-space. Figure 5 displays the efficient frontier together with the industry assets, where we have materialized the **MV** and **AV** funds.

There are four other portfolios represented on the efficient frontier: MKT1, MKT2, MKT1 $\perp$, and MKT2 $\perp$. We start by discussing the portfolios MKT1 and MKT2. To understand these portfolios, let $m_t$ be the $n$-dimensional vector of market capitalizations of the $n = 11$ industry assets at date $t$ and $r_t$ be the $n$-dimensional vector of asset returns. Construct the portfolio weight series $\tilde{w}_{p,t}$ and the portfolio return series $\tilde{r}_{p,t}$ and $\bar{r}_{p,t}$ as follows

$$\begin{aligned} \tilde{w}_{p,t} &= \frac{m_t}{\mathbf{1}^\top m_t} \quad \text{and} \quad \tilde{r}_{p,t} = \tilde{w}_{p,t-1}^\top r_t \\ \bar{w}_p &= \frac{1}{T}\sum_{t=1}^{T} \tilde{w}_{p,t} \quad \text{and} \quad \bar{r}_{p,t} = \bar{w}_p^\top r_t. \end{aligned} \tag{24}$$

Now, the MKT1 portfolio is the efficient portfolio with targeted expected return and degree of non-normality of the return series $\tilde{r}_{p,t}$. Likewise, the MKT2 portfolio is the efficient portfolio with targeted expected return and degree of non-normality of the return series $\bar{r}_{p,t}$. They both measure the market portfolio in our calibrated economy with the 11 industry assets. These portfolios will be subsequently used to illustrate the asset pricing implications of our portfolio optimization model.

## 6 Dynamic Illustration of the Optimal Portfolio Strategy

Our aim in this section is to illustrate how our asset allocation model, despite being a static model, can be applicable in a dynamic context. For that, we start by considering two simple rolling portfolio strategies defined as follows. For the first strategy, at any date $t-1$, the investor holds the portfolio $\tilde{w}_{p,t-1}$ for one period, and gets the return $\tilde{r}_{p,t} = \tilde{w}_{p,t-1}^\top r_t$ at date $t$. She then repeats this over time. We refer to this strategy as the "value-weighted" portfolio strategy. For the second strategy, the investor at any date $t-1$, holds the portfolio $\bar{w}_{p,t-1} = \frac{1}{h}\sum_{j=1}^{h} \tilde{w}_{p,t-j}$ for one period, where $h$ is the rolling window, and gets the return $\bar{r}_{p,t} = \bar{w}_{p,t-1}^\top r_t$ at date $t$. She then repeats this over time. We refer to this strategy as the "constant-average" value-weighted portfolio strategy.

The two previous strategies are suboptimal. We now consider their optimal counterparts. At date $t-1$, the investor uses available asset returns data over the rolling period from $t-h$ to $t-1$ to estimate parameters of the multivariate normal-exponential model of asset returns. Given these parameters, the investor computes the optimal portfolio targeting the mean and the degree of non-normality of each of the two suboptimal portfolios. In the optimal strategy, instead of holding the suboptimal portfolio for the next period, the investor holds the optimal portfolio that has the same characteristics as the suboptimal portfolio. Correspondingly, we refer to the first optimal strategy as the "optimal value-weighted" portfolio strategy and to second as the optimal "constant-average value-weighted" portfolio strategy.

We would like to compare the performance of these suboptimal and optimal rolling portfolio strategies in terms of volatility, Sharpe ratio and certainty equivalent. Here, the investor makes her investment decision based on observed past data and introduces some dynamics by repeating the static decision process as new information arrives. We use a rolling window of 44 years of monthly observations starting from January 1970. Figure 6 shows the dynamics of expected return and degree of non-normality of the two suboptimal strategies as well as for their corresponding optimal strategies. It also shows the dynamics of volatility and the Sharpe ratio of those strategies and of their corresponding optimal strategies. The volatility of the optimal portfolio strategy is always far below the suboptimal counterpart, which means that the optimal portfolio strategy allows to reduce the total risk borne by the investor. The Sharpe ratio further confirms the superiority of the optimal strategy; indeed, the reward-to-risk ratio is higher for the optimal portfolio strategy than the suboptimal counterpart. The two optimal strategies yield the same reward-to-risk ratio, even though they differ in terms of average return and degree of non-normality.

Using the endogenous preference parameters computed under the optimal strategies, we compute the dynamic evolution of the certainty equivalent of the four portfolio strategies (the two optimal strategies and their suboptimal counterparts). Figure 7 (left panel) plots the results that confirm the superiority of the optimal strategies for which the certainty equivalents are positive, over the suboptimal counterparts for which the certainty equivalents are negative. The right panel of Figure 7 represents the cumulative nominal returns of the different strategies. In this panel, the optimal counterpart targets the same level of risk as the suboptimal strategy and it uses the optimal Sharpe ratio in order to infer the expected return. We invest \$1 in January 1970 and plot the cumulative returns of each strategy. The optimal strategies realize relatively huge gains compared with their suboptimal counterparts. By the end of the sample period in November 2020, the gains from the optimal counterparts cumulate to around \$1,000 versus about \$200 for the suboptimal strategy.

# 7 Asset Pricing Implications: Multiple Investors' Setting

We now extend the previous setting and assume $k$ investors in our single-period economy, denoted $j = 1, 2, \ldots, k$, where all investors care about minimizing their portfolio variance for a given minimum level of expected return and degree of non-normality, and that they all agree on the asset returns distribution and parameters. Therefore, the optimal solution to the portfolio choice problem for investor $j$ is

$$w^{(j)} = \lambda_1^{(j)} \Sigma^{-1} \left( \mu - \mathbf{1} r_f \right) + \lambda_2^{(j)} \Sigma^{-1} \left( \sigma \odot \delta \right), \tag{25}$$

where the multipliers $\lambda_1^{(j)}$ and $\lambda_2^{(j)}$ are endogenous investor's preference parameters, which depend on the investor's targeted minimum level of expected return $\mu_w^{(j)}$ and degree of non-normality $\sigma_w^{(j)} \delta_w^{(j)}$ as shown in equation (16).

### 7.1 Characterization of the market portfolio

We further assume that each investor $j$ has initial wealth $W^{(j)}$. Denote by $W$ the total market capitalization and let $w$ be the market portfolio. Aggregating asset demands across investors yields

$$\sum_{j=1}^{k} w^{(j)} W^{(j)} = wW \quad \text{with} \quad W = \sum_{j=1}^{k} W^{(j)}, \tag{26}$$

implying that

$$w = \lambda_1 \Sigma^{-1} \left( \mu - \mathbf{1} r_f \right) + \lambda_2 \Sigma^{-1} \left( \sigma \odot \delta \right), \tag{27}$$

where

$$\lambda_1 = \sum_{j=1}^{k} \pi^{(j)} \lambda_1^{(j)} \quad \text{and} \quad \lambda_2 = \sum_{j=1}^{k} \pi^{(j)} \lambda_2^{(j)}, \quad \text{and} \quad \pi^{(j)} = \frac{W^{(j)}}{W}. \tag{28}$$

Equation (27) clearly shows that at equilibrium the market portfolio is efficient and interpretable as the optimal portfolio of a representative investor with preference parameters $\lambda_1$ and $\lambda_2$ who, similar to each individual investor, cares about minimizing her portfolio variance for given level of expected return and degree of non-normality, and agrees on the same asset returns distribution and parameters. Equation (28) shows how preference parameters of the representative investor are obtained from those of the individual investors by aggregation.

Finally observe that equation (28) is also equivalent to

$$\begin{aligned} \lambda_1 &= \frac{A}{AC - B^2} \left( \sum_{j=1}^{k} \pi^{(j)} \mu_w^{(j)} - r_f \right) - \frac{B}{AC - B^2} \sum_{j=1}^{k} \pi^{(j)} \sigma_w^{(j)} \delta_w^{(j)} \\ \lambda_2 &= -\frac{B}{AC - B^2} \left( \sum_{j=1}^{k} \pi^{(j)} \mu_w^{(j)} - r_f \right) + \frac{C}{AC - B^2} \sum_{j=1}^{k} \pi^{(j)} \sigma_w^{(j)} \delta_w^{(j)}. \end{aligned} \tag{29}$$

Thus, the representative investor's targeted minimum level of expected return and degree of non-normality are, respectively, given by $\mu_w = \sum_{j=1}^{k} \pi^{(j)} \mu_w^{(j)}$ and $\sigma_w \delta_w = \sum_{j=1}^{k} \pi^{(j)} \sigma_w^{(j)} \delta_w^{(j)}$, i.e., they are weighted averages of similar individual investor quantities, with each individual $j$'s weight being her share of market capitalisation $\pi^{(j)}$.

### 7.2 Characterization of the orthogonal portfolio

Let us consider the efficient portfolio that has zero covariance with the market portfolio $w$ and has the opposite degree of non-normality. Observe that any efficient portfolio may be written as a linear combination of two other efficient portfolios. Formally, we are looking for the portfolio $w^*$ such that $w^* = c_1 w + c_2 \Sigma^{-1} (\sigma \odot \delta)$ and the coefficients $c_1$ and $c_2$ are chosen to satisfy the following two conditions: $w^\top \Sigma w^* = 0$ and $(\sigma \odot \delta)^\top w^* = -\sigma_w \delta_w$. We obtain

$$c_1 = \frac{1}{\sigma_w^2} \left( \frac{A}{(\sigma_w \delta_w)^2} - \frac{1}{\sigma_w^2} \right)^{-1} \quad \text{and} \quad c_2 = -\frac{1}{\sigma_w \delta_w} \left( \frac{A}{(\sigma_w \delta_w)^2} - \frac{1}{\sigma_w^2} \right)^{-1}. \tag{30}$$

Thus

$$w^* = \left( \frac{A}{(\sigma_w \delta_w)^2} - \frac{1}{\sigma_w^2} \right)^{-1} \left( \frac{w}{\sigma_w^2} - \frac{\Sigma^{-1} (\sigma \odot \delta)}{\sigma_w \delta_w} \right). \tag{31}$$

From equation (31), we can compute the variance of the orthogonal portfolio $\sigma_{w^*}^2 = w^{*\top} \Sigma w^*$, and show that

$$\sigma_{w^*}^2 = \left( \frac{A}{(\sigma_w \delta_w)^2} - \frac{1}{\sigma_w^2} \right)^{-1}. \tag{32}$$

The portfolios MKT1 $\perp$ and MKT2 $\perp$ represented on the efficient frontier in Figure 5 are the orthogonal counterparts to the portfolios MKT1 and MKT2, respectively.

### 7.3 Characterization of asset risk premia

We recall that any asset $i = 1, 2, \ldots, n$ can be viewed as a portfolio $u_i$, where $u_i$ denotes a unit vector of order $n$, that is, $u_i$ has unity in its $i$-th position and zeros elsewhere. From equation (31), we compute the covariance of asset $i$ with the orthogonal portfolio $\sigma_{iw^*} = u_i^\top \Sigma w^*$, and show that

$$\sigma_{iw^*} = \left( \frac{A}{(\sigma_w \delta_w)^2} - \frac{1}{\sigma_w^2} \right)^{-1} \left( \frac{\sigma_{iw}}{\sigma_w^2} - \frac{\sigma_i \delta_i}{\sigma_w \delta_w} \right), \tag{33}$$

where $\sigma_{iw}$ is the covariance between asset $i$ and the market portfolio. Equations (33) and (32) together imply that

$$\beta_{iw^*} = \beta_{iw} - \frac{\sigma_i \delta_i}{\sigma_w \delta_w} \text{ or, equivalently, } \beta_{iw^*} - \beta_{iw} = -\frac{\sigma_i \delta_i}{\sigma_w \delta_w}, \tag{34}$$

where $\beta_{iw} \equiv \dfrac{\sigma_{iw}}{\sigma_w^2}$ and $\beta_{iw^*} \equiv \dfrac{\sigma_{iw^*}}{\sigma_{w^*}^2}$ define the asset's betas on the market and orthogonal portfolios, respectively. We prove in the appendix that in equilibrium, the risk premium of asset $i$ is given by

$$\mu_i - r_f = \beta_{iw} \left( \mu_w - r_f \right) + \beta_{iw^*} \left( \mu_{w^*} - r_f \right). \tag{35}$$

Notice that $\beta_{iw}$ is the Capital Asset Pricing Model (CAPM) beta as usually understood, whereas $\beta_{iw^*}$ adds to it a component reflecting the asset's non-normality, that is, the beta spread of equation (34) is a measure of non-normality. Thus, in equilibrium, the asset risk premium reflects asset non-normality. In particular, as we can see, negative asymmetry ($\delta_i < 0$) in asset returns leads to higher beta with respect to the orthogonal portfolio, i.e., higher $\beta_{iw^*}$ and, thus, a higher asset risk premium. Finally, we have shown that in an economy where asset returns are generated by a multivariate normal-exponential model, asset risk premia are compensation for two sources of risk: exposure to the market portfolio and degree of non-normality. These two risks can be measured as betas from a two-factor linear regression model relating the asset excess return to excess returns on the market portfolio and an orthogonal portfolio.

Figure 8 shows the risk exposures of U.S. industry portfolios with respect to the market and the orthogonal portfolio. In the left panels, the market portfolio is assumed to be the MKT1 portfolio on the efficient frontier while in the right panel is assumed to be MKT2. In each panel of the figure is displayed the output of the regression of asset risk premia onto the corresponding risk measures (i.e., the factor betas). Interestingly, regardless of the assumed measure of the market portfolio, these output clearly evidence that in our economy, contrary to the CAPM predictions, asset risk premiums are better explained by their exposure to the orthogonal portfolio and shown by equation (34) to reflect the asset degree of non-normality. The betas on the orthogonal portfolio capture more than 60% of the variation in expected returns while the CAPM betas only explain at most 12% on this variation. Therefore, our results suggest that in an economy where asset returns are non-normally distributed and where investors have non-normality concerns when making investment decisions, the degree of asset non-normality is key to explaining cross-sectional differences in expected returns.

# 8 Conclusion

This paper investigates the implications of assets returns non-normality and investors' non-normality concerns for asset allocation and asset pricing. Non-normality concerns arise from adding a linear non-normality constraint to an otherwise standard portfolio optimization problem. We assume a multivariate normal-exponential model for the asset returns and develop a procedure for model parameter estimation using a GMM with exact moment conditions. We solve the optimal portfolio allocation problem for an investor with targeted minimum expected return and degree of non-normality. The optimal allocation boils down to a three-fund separation formula where the investor holds the risk-free asset, the mean-variance fund and a non-normality-variance fund. We further characterize non-participation to the risky assets market and highlight the welfare losses that arise from not accounting for non-normality.

Our model also shows that accounting for non-normality could enable the better capture of systematic risks embedded in assets compared with the CAPM predictions and, thus, improve asset pricing and capital budgeting. Finally, in a dynamic context, we illustrate the superiority of our optimal strategies over trivial rolling portfolio strategies. Possible extensions of this model that could be worth considering in future work would be to consider a more sophisticated model where the assets returns are conditionally multivariate normal-exponential and the parameters move through time, and embed this setting into a dynamic consumption-investment problem where intertemporal hedging demand pertaining to non-normality can be characterized.

# Appendix

**Solution to the portfolio optimization problem** Formally, the problem is stated as follows:

$$\min_{w} \frac{1}{2} w^{\top} \Sigma w \qquad \text{subject to} \qquad w^{\top}\left(\mu - \mathbf{1} r_f\right) \geq \mu_w - r_f, \qquad w^{\top}(\sigma \odot \delta) = \sigma_w \delta_w, \tag{A.1}$$

where $\mu_w$ and $\sigma_w \delta_w$ are given.

The Lagrangian of this quadratic program is given by:

$$L\left(w, \lambda_1, \lambda_2\right) = \frac{1}{2} w^{\top} \Sigma w - \lambda_1 \left(w^{\top}\left(\mu - \mathbf{1} r_f\right) - \left(\mu_w - r_f\right)\right) - \lambda_2 \left(w^{\top}(\sigma \odot \delta) - \sigma_w \delta_w\right)$$

From which the following Karush Kuhn Tucker (KKT) conditions can be deduced:

$$0 = \Sigma w - \lambda_1 \left(\mu - \mathbf{1} r_f\right) - \lambda_2 (\sigma \odot \delta) \tag{A.2}$$

$$0 = \lambda_1 \left(w^{\top}\left(\mu - \mathbf{1} r_f\right) - \left(\mu_w - r_f\right)\right) \tag{A.3}$$

$$0 \leq w^{\top}\left(\mu - \mathbf{1} r_f\right) - \left(\mu_w - r_f\right) \tag{A.4}$$

$$0 \leq \lambda_1 \tag{A.5}$$

$$0 = w^{\top}(\sigma \odot \delta) - \sigma_w \delta_w \tag{A.6}$$

The first condition is equivalent to

$$w = \lambda_1 \Sigma^{-1}\left(\mu - \mathbf{1} r_f\right) + \lambda_2 \Sigma^{-1}(\sigma \odot \delta)$$

.

- **Corner solution:** There is a corner solution when the expected return constraint is binding, i.e., $0 = w^{\top}\left(\mu - \mathbf{1} r_f\right) - \left(\mu_w - r_f\right)$. Therefore, $\lambda_1 \geq 0$. Substituting out the optimal portfolio $w$ by its expression in terms of the multipliers, the two constraints lead to the following system:

$$\lambda_1 C + \lambda_2 B = \mu_w - r_f$$
$$\lambda_1 B + \lambda_2 A = \sigma_w \delta_w.$$

It follows that:

$$\begin{aligned}
\lambda_1 &= \frac{A}{AC - B^2}\left(\mu_w - r_f\right) - \frac{B}{AC - B^2}\sigma_w \delta_w \\
\lambda_2 &= -\frac{B}{AC - B^2}\left(\mu_w - r_f\right) + \frac{C}{AC - B^2}\sigma_w \delta_w
\end{aligned} \tag{A.7}$$

where

$$A = (\sigma \odot \delta)^\top \Sigma^{-1} (\sigma \odot \delta), \quad B = \left(\mu - \mathbf{1} r_f\right)^\top \Sigma^{-1} (\sigma \odot \delta), \quad C = \left(\mu - \mathbf{1} r_f\right)^\top \Sigma^{-1} \left(\mu - \mathbf{1} r_f\right).$$

Notice that the nonnegativity of $\lambda_1$ is equivalent to $\mu_w \geq r_f + \frac{B}{A}\sigma_w \delta_w$, which is the condition on the targets that must hold for a corner solution to be valid.

- **Interior solution:** There is an interior solution when the expected return constraint is not binding, i.e., $0 < w^\top \left(\mu - \mathbf{1} r_f\right) - \left(\mu_w - r_f\right)$. Therefore, $\lambda_1 = 0$. It follows that $w = \lambda_2 \Sigma^{-1} (\sigma \odot \delta)$. Substituting out the optimal portfolio $w$ by its expression in term of the second multiplier, the non-normality constraint leads to

$$\lambda_2 = \frac{\sigma_w \delta_w}{A}.$$

Notice that the non-binding expected return constraint is equivalent to $\mu_w < r_f + \frac{B}{A}\sigma_w \delta_w$, which is the condition on the targets that must hold for an interior solution to be valid.

**Characterization of asset risk premia** From equation (27) of the main body of the paper, we can express the vector of asset risk premia as follows:

$$\mu - \mathbf{1} r_f = \frac{1}{\lambda_1} \Sigma w - \frac{\lambda_2}{\lambda_1} \left( \sigma \odot \delta \right). \tag{A.8}$$

We pre-multiply equation (A.8) by the market portfolio $w^\top$, then by the orthogonal portfolio $w^{*\top}$, and by the unit vector $u_i^\top$ to obtain the following system:

$$\begin{cases} \mu_w - r_f &= \dfrac{1}{\lambda_1}\sigma_w^2 - \dfrac{\lambda_2}{\lambda_1}\sigma_w\delta_w \\ \mu_{w^*} - r_f &= \dfrac{\lambda_2}{\lambda_1}\sigma_w\delta_w \\ \mu_i - r_f &= \dfrac{1}{\lambda_1}\sigma_{iw} - \dfrac{\lambda_2}{\lambda_1}\sigma_i\delta_i. \end{cases} \tag{A.9}$$

From the first two equations of the system, the representative investor endogenous preference parameters can be expressed as follows:

$$\frac{1}{\lambda_1} = \frac{\left(\mu_w - r_f\right) + \left(\mu_{w^*} - r_f\right)}{\sigma_w^2} \quad \text{and} \quad \frac{\lambda_2}{\lambda_1} = \frac{\mu_{w^*} - r_f}{\sigma_w \delta_w}. \tag{A.10}$$

Using their expressions in equation (A.10), substitute out the representative investor endogenous preference parameters in the third equation of system (A.9) to obtain

$$\mu_i - r_f = \frac{\sigma_{iw}}{\sigma_w^2}\left(\mu_w - r_f\right) + \left(\frac{\sigma_{iw}}{\sigma_w^2} - \frac{\sigma_i\delta_i}{\sigma_w\delta_w}\right)\left(\mu_{w^*} - r_f\right). \tag{A.11}$$

Equation (A.11) is equivalent to equation (35) of the main body of the paper.

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| | A. Sample moments | | | | B. Model-implied skewness | | | | | C. Model-implied kurtosis | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | mean(%) | std(%) | skew | kurt | Sp1 | Sp2 | Sp3 | Sp4 | Sp5 | Sp1 | Sp2 | Sp3 | Sp4 | Sp5 |
| Fd | 0.97 | 4.77 | 0.030 | 8.73 | 0.030 | 0.006 | 0.192 | 0.006 | 0.077 | 3.02 | 3.00 | 3.26 | 3.00 | 3.08 |
| Cl | 0.96 | 6.15 | 0.257 | 7.33 | 0.257 | 0.248 | 0.482 | 0.303 | 0.391 | 3.39 | 3.37 | 3.90 | 3.48 | 3.68 |
| Bm | 1.01 | 6.96 | 0.315 | 8.81 | 0.315 | 0.209 | 0.567 | 0.235 | 0.376 | 3.51 | 3.30 | 4.12 | 3.35 | 3.65 |
| Mn | 1.02 | 7.29 | 0.050 | 6.32 | 0.050 | 0.002 | 0.204 | 0.002 | 0.062 | 3.04 | 3.00 | 3.29 | 3.00 | 3.06 |
| Oi | 0.96 | 6.35 | 0.234 | 7.54 | 0.234 | 0.033 | 0.365 | 0.029 | 0.149 | 3.34 | 3.02 | 3.62 | 3.02 | 3.19 |
| Hw | 1.20 | 7.35 | 0.097 | 7.42 | 0.097 | 0.081 | 0.271 | 0.083 | 0.201 | 3.11 | 3.08 | 3.42 | 3.09 | 3.28 |
| Ch | 1.22 | 8.59 | 0.396 | 8.82 | 0.396 | 0.352 | 0.607 | 0.369 | 0.472 | 3.69 | 3.59 | 4.22 | 3.63 | 3.88 |
| Bx | 1.08 | 6.14 | 0.140 | 8.38 | 0.140 | 0.082 | 0.304 | 0.082 | 0.211 | 3.17 | 3.09 | 3.49 | 3.08 | 3.30 |
| Rt | 1.07 | 5.98 | 0.038 | 8.70 | 0.038 | 0.068 | 0.191 | 0.070 | 0.187 | 3.03 | 3.07 | 3.26 | 3.07 | 3.25 |
| Bk | 1.15 | 7.06 | 0.006 | 7.80 | 0.006 | 0.006 | 0.153 | 0.006 | 0.061 | 3.00 | 3.00 | 3.19 | 3.00 | 3.06 |
| Ot | 0.72 | 7.32 | 0.012 | 6.83 | 0.012 | 0.006 | 0.224 | 0.006 | 0.124 | 3.01 | 3.00 | 3.32 | 3.00 | 3.15 |

Table 1: **Summary statistics and model-implied higher-order moments**
This table shows on Panel A the sample moments (mean, standard deviation, skewness, and kurtosis) of the industry portfolios and, on Panels B and C, the model-implied skewness and model-implied kurtosis, respectively, using the GMM estimates for the different specifications as described in Section 3. Our sample is made of monthly returns data from July 1926 to November 2020 and covers 11 industries listed in the first column of the table.

| | Sp1 | Sp2 | Sp3 | Sp4 | Sp5 | Sp1 | Sp2 | Sp3 | Sp4 | Sp5 | Sp1 | Sp2 | Sp3 | Sp4 | Sp5 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | A. Estimates of $\mu \times 100$ with standard errors | | | | | B. Estimates of $a \times 100$ with standard errors | | | | | C. Implied values of $\sigma \times 100$ and $\delta$ | | | | |
| Fd | 0.97 | 0.95 | 0.96 | 0.95 | 0.86 | 1.17 | 0.68 | 2.24 | 0.70 | 1.79 | 4.76 | 4.76 | 4.89 | 4.89 | 5.30 |
| | **0.15** | **0.19** | **0.15** | **0.20** | **0.16** | **5.48** | **4.46** | **1.59** | **3.96** | **1.89** | **0.25** | **0.14** | **0.46** | **0.14** | **0.34** |
| Cl | 0.96 | 0.91 | 0.93 | 0.88 | 0.70 | 3.10 | 3.07 | 3.90 | 3.35 | 3.84 | 6.15 | 6.15 | 6.27 | 6.29 | 6.61 |
| | **0.20** | **0.21** | **0.20** | **0.21** | **0.22** | **1.48** | **1.39** | **1.02** | **1.16** | **1.03** | **0.50** | **0.50** | **0.62** | **0.53** | **0.58** |
| Bm | 1.01 | 1.01 | 0.98 | 0.99 | 0.78 | 3.75 | 3.27 | 4.70 | 3.52 | 4.50 | 6.95 | 6.95 | 7.15 | 7.19 | 7.85 |
| | **0.22** | **0.22** | **0.22** | **0.22** | **0.22** | **1.79** | **1.93** | **1.26** | **1.69** | **1.18** | **0.54** | **0.47** | **0.66** | **0.49** | **0.57** |
| Mn | 1.02 | 1.00 | 0.99 | 1.00 | 0.83 | 2.13 | 0.75 | 3.48 | 0.77 | 2.49 | 7.29 | 7.29 | 7.45 | 7.47 | 7.93 |
| | **0.23** | **0.31** | **0.23** | **0.32** | **0.24** | **4.58** | **5.39** | **1.79** | **4.76** | **1.96** | **0.29** | **0.10** | **0.47** | **0.10** | **0.31** |
| Oi | 0.96 | 0.98 | 0.94 | 0.98 | 0.81 | 3.10 | 1.61 | 3.69 | 1.59 | 2.96 | 6.34 | 6.34 | 6.51 | 6.54 | 7.03 |
| | **0.19** | **0.21** | **0.19** | **0.22** | **0.19** | **1.32** | **3.23** | **0.98** | **3.08** | **1.34** | **0.49** | **0.25** | **0.57** | **0.24** | **0.42** |
| Hw | 1.20 | 1.20 | 1.18 | 1.19 | 1.01 | 2.67 | 2.52 | 3.85 | 2.60 | 3.69 | 7.34 | 7.34 | 7.50 | 7.51 | 7.93 |
| | **0.24** | **0.26** | **0.24** | **0.26** | **0.25** | **3.86** | **3.68** | **2.07** | **3.41** | **2.12** | **0.36** | **0.34** | **0.51** | **0.35** | **0.46** |
| Ch | 1.22 | 1.25 | 1.19 | 1.23 | 0.99 | 5.01 | 4.81 | 5.87 | 4.99 | 5.75 | 8.59 | 8.59 | 8.74 | 8.76 | 9.30 |
| | **0.29** | **0.28** | **0.29** | **0.28** | **0.30** | **2.23** | **1.80** | **1.84** | **1.82** | **1.74** | **0.58** | **0.56** | **0.67** | **0.57** | **0.62** |
| Bx | 1.08 | 1.07 | 1.06 | 1.06 | 0.92 | 2.52 | 2.12 | 3.33 | 2.15 | 3.14 | 6.13 | 6.13 | 6.24 | 6.25 | 6.64 |
| | **0.19** | **0.20** | **0.19** | **0.20** | **0.18** | **2.66** | **3.26** | **1.73** | **3.09** | **1.80** | **0.41** | **0.35** | **0.53** | **0.34** | **0.47** |
| Rt | 1.07 | 1.05 | 1.05 | 1.04 | 0.90 | 1.60 | 1.93 | 2.80 | 2.00 | 3.01 | 5.98 | 5.98 | 6.13 | 6.13 | 6.63 |
| | **0.19** | **0.21** | **0.19** | **0.21** | **0.20** | **5.77** | **3.37** | **2.03** | **3.09** | **1.72** | **0.27** | **0.32** | **0.46** | **0.33** | **0.45** |
| Bk | 1.15 | 1.11 | 1.13 | 1.10 | 0.96 | 1.02 | 0.99 | 3.13 | 1.04 | 2.56 | 7.05 | 7.05 | 7.37 | 7.39 | 8.22 |
| | **0.22** | **0.30** | **0.22** | **0.30** | **0.23** | **19.03** | **5.94** | **2.08** | **5.27** | **2.33** | **0.14** | **0.14** | **0.42** | **0.14** | **0.31** |
| Ot | 0.72 | 0.69 | 0.68 | 0.68 | 0.45 | 1.32 | 1.07 | 3.63 | 1.12 | 3.22 | 7.32 | 7.32 | 7.53 | 7.56 | 8.14 |
| | **0.24** | **0.31** | **0.24** | **0.31** | **0.24** | **12.81** | **5.97** | **1.84** | **5.31** | **1.92** | **0.18** | **0.15** | **0.48** | **0.15** | **0.40** |

| | Fd | Cl | Bm | Mn | Oi | Hw | Ch | Bx | Rt | Bk | Ot |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | D. GMM Specification 1: Estimate of $X \times 100$ with standard errors | | | | | | | | | | |
| Fd | 4.62 | | | | | | | | | | |
| | **1.30** | | | | | | | | | | |
| Cl | 3.24 | 4.20 | | | | | | | | | |
| | **2.89** | **1.73** | | | | | | | | | |
| Bm | 4.44 | 1.17 | 3.63 | | | | | | | | |
| | **3.30** | **3.48** | **1.76** | | | | | | | | |
| Mn | 3.47 | 1.60 | 2.60 | 5.22 | | | | | | | |
| | **2.30** | **2.14** | **1.90** | **0.80** | | | | | | | |
| Oi | 3.02 | -0.01 | 1.22 | 1.95 | 4.02 | | | | | | |
| | **2.95** | **2.95** | **2.81** | **1.72** | **1.34** | | | | | | |
| Hw | 3.77 | 1.30 | 2.03 | 0.69 | 0.44 | 5.11 | | | | | |
| | **2.70** | **1.92** | **1.10** | **0.43** | **1.10** | **0.39** | | | | | |
| Ch | 3.97 | 0.86 | 2.13 | 1.07 | -0.56 | 2.82 | 4.27 | | | | |
| | **4.99** | **4.18** | **3.44** | **1.91** | **2.42** | **1.57** | **1.69** | | | | |
| Bx | 3.79 | 1.01 | 1.42 | 0.72 | 0.33 | 0.75 | 0.51 | 3.52 | | | |
| | **2.32** | **2.04** | **1.15** | **0.33** | **0.65** | **0.18** | **0.63** | **0.16** | | | |
| Rt | 4.44 | 1.86 | 1.53 | -0.12 | 0.24 | 0.68 | 1.02 | 0.31 | 2.44 | | |
| | **1.85** | **1.11** | **2.09** | **1.43** | **2.40** | **0.75** | **3.22** | **0.52** | **3.24** | | |
| Bk | 4.86 | 1.83 | 2.68 | -0.15 | 1.11 | 0.51 | 1.35 | 0.48 | -0.82 | 3.22 | |
| | **4.48** | **8.33** | **10.07** | **4.76** | **10.34** | **1.86** | **14.05** | **1.05** | **22.74** | **29.21** | |
| Ot | 4.58 | 2.23 | 2.67 | 0.36 | 0.90 | 0.68 | 1.75 | -0.10 | -1.29 | -1.48 | 3.24 |
| | **3.24** | **5.18** | **6.58** | **3.47** | **7.00** | **1.48** | **9.52** | **0.97** | **17.53** | **63.03** | **49.74** |

Table 2: **GMM parameter estimates and implied parameters**

This table presents estimates of the model parameters obtained by GMM either directly or indirectly through reparameterization as described in Section 2. The standard error of each estimate is given in bold below it. On Panels A, B and C, each column represents the estimates from a given GMM specification: **Sp1** considers means, variances, covariances and skewness; **Sp2** adds coskewness to the moments in **Sp1**; **Sp3** adds kurtoses to the moments in **Sp1**, **Sp4** adds kurtoses to the moments in **Sp2**; and **Sp5** considers all moments together. Panel D presents the GMM estimates of $X$ for the specification **Sp1**.

| | Fd | Cl | Bm | Mn | Oi | Hw | Ch | Bx | Rt | Bk | Ot |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | A. GMM Specification 2: Estimate of $X \times 100$ with standard errors | | | | | | | | | | |
| Fd | 4.72 | | | | | | | | | | |
| | **0.56** | | | | | | | | | | |
| Cl | 3.51 | 4.01 | | | | | | | | | |
| | **2.47** | **1.77** | | | | | | | | | |
| Bm | 4.81 | 1.01 | 3.67 | | | | | | | | |
| | **2.44** | **3.03** | **1.30** | | | | | | | | |
| Mn | 3.82 | 2.21 | 3.17 | 4.80 | | | | | | | |
| | **0.87** | **2.24** | **3.26** | **3.04** | | | | | | | |
| Oi | 3.50 | 0.54 | 1.86 | 1.83 | 4.28 | | | | | | |
| | **1.37** | **0.85** | **0.58** | **0.67** | **0.33** | | | | | | |
| Hw | 3.99 | 1.06 | 1.94 | 0.85 | 0.35 | 5.09 | | | | | |
| | **2.08** | **1.43** | **0.71** | **0.82** | **0.38** | **0.52** | | | | | |
| Ch | 4.44 | 0.42 | 2.07 | 1.84 | -0.19 | 2.53 | 4.07 | | | | |
| | **3.84** | **4.32** | **3.26** | **4.82** | **1.56** | **1.33** | **3.35** | | | | |
| Bx | 4.03 | 0.93 | 1.46 | 0.81 | 0.35 | 0.71 | 0.42 | 3.52 | | | |
| | **1.72** | **1.27** | **0.51** | **0.49** | **0.29** | **0.27** | **0.68** | **0.16** | | | |
| Rt | 4.47 | 1.39 | 1.15 | -0.02 | -0.15 | 0.72 | 0.63 | 0.35 | 2.77 | | |
| | **1.49** | **0.93** | **0.33** | **0.66** | **0.22** | **0.15** | **1.30** | **0.21** | **0.39** | | |
| Bk | 4.87 | 1.61 | 2.44 | -0.52 | 0.51 | 0.72 | 1.26 | 0.58 | 0.25 | 3.67 | |
| | **1.02** | **1.45** | **2.41** | **4.57** | **1.30** | **0.67** | **6.99** | **0.83** | **3.07** | **5.20** | |
| Ot | 4.65 | 2.17 | 2.58 | -0.11 | 0.39 | 0.90 | 1.86 | 0.00 | -0.25 | -0.54 | 3.82 |
| | **1.11** | **1.72** | **2.75** | **5.00** | **1.37** | **0.67** | **7.55** | **0.90** | **3.49** | **12.75** | **8.68** |
| | C. GMM Specification 4: Estimate of $X \times 100$ with standard errors | | | | | | | | | | |
| Fd | 4.84 | | | | | | | | | | |
| | **0.50** | | | | | | | | | | |
| Cl | 3.36 | 4.13 | | | | | | | | | |
| | **2.40** | **1.67** | | | | | | | | | |
| Bm | 4.64 | 0.87 | 4.13 | | | | | | | | |
| | **2.32** | **2.79** | **1.25** | | | | | | | | |
| Mn | 3.71 | 2.30 | 3.09 | 5.15 | | | | | | | |
| | **0.80** | **2.18** | **2.85** | **2.65** | | | | | | | |
| Oi | 3.40 | 0.63 | 1.83 | 1.84 | 4.64 | | | | | | |
| | **1.22** | **0.66** | **0.64** | **0.60** | **0.34** | | | | | | |
| Hw | 3.86 | 1.03 | 1.85 | 1.02 | 0.44 | 5.38 | | | | | |
| | **1.90** | **1.17** | **0.66** | **0.74** | **0.32** | **0.52** | | | | | |
| Ch | 4.28 | 0.22 | 1.82 | 2.14 | 0.01 | 2.48 | 4.41 | | | | |
| | **3.50** | **3.71** | **2.77** | **4.32** | **1.14** | **1.28** | **3.24** | | | | |
| Bx | 3.91 | 0.96 | 1.47 | 0.89 | 0.44 | 0.77 | 0.51 | 3.76 | | | |
| | **1.57** | **0.96** | **0.46** | **0.42** | **0.22** | **0.24** | **0.63** | **0.22** | | | |
| Rt | 4.34 | 1.42 | 1.23 | 0.09 | -0.06 | 0.79 | 0.74 | 0.41 | 3.14 | | |
| | **1.39** | **0.70** | **0.36** | **0.69** | **0.17** | **0.15** | **1.37** | **0.27** | **0.50** | | |
| Bk | 4.73 | 1.75 | 2.46 | -0.36 | 0.56 | 0.85 | 1.46 | 0.61 | 0.28 | 4.28 | |
| | **0.96** | **1.49** | **2.19** | **3.92** | **0.79** | **0.63** | **6.19** | **0.95** | **2.88** | **4.24** | |
| Ot | 4.52 | 2.26 | 2.59 | 0.04 | 0.45 | 1.01 | 2.02 | 0.05 | -0.19 | -0.50 | 4.24 |
| | **1.05** | **1.75** | **2.46** | **4.28** | **0.84** | **0.63** | **6.69** | **1.01** | **3.29** | **9.97** | **7.01** |

| | Fd | Cl | Bm | Mn | Oi | Hw | Ch | Bx | Rt | Bk | Ot |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | B. GMM Specification 3: Estimate of $X \times 100$ with standard errors | | | | | | | | | | |
| Fd | 4.35 | | | | | | | | | | |
| | **0.71** | | | | | | | | | | |
| Cl | 2.27 | 4.35 | | | | | | | | | |
| | **1.34** | **0.47** | | | | | | | | | |
| Bm | 3.31 | 1.17 | 4.09 | | | | | | | | |
| | **1.38** | **0.57** | **0.35** | | | | | | | | |
| Mn | 2.47 | 1.23 | 2.13 | 5.59 | | | | | | | |
| | **1.41** | **1.04** | **0.65** | **0.28** | | | | | | | |
| Oi | 2.15 | 0.01 | 1.22 | 1.73 | 4.43 | | | | | | |
| | **1.25** | **0.50** | **0.39** | **0.46** | **0.31** | | | | | | |
| Hw | 2.74 | 1.08 | 1.76 | 0.80 | 0.36 | 5.37 | | | | | |
| | **1.60** | **0.86** | **0.63** | **0.24** | **0.44** | **0.47** | | | | | |
| Ch | 2.55 | 0.76 | 2.01 | 0.89 | -0.27 | 2.63 | 4.79 | | | | |
| | **2.15** | **0.83** | **0.60** | **0.64** | **0.45** | **0.93** | **0.38** | | | | |
| Bx | 2.99 | 1.05 | 1.43 | 0.76 | 0.44 | 0.79 | 0.62 | 3.73 | | | |
| | **1.29** | **0.54** | **0.43** | **0.16** | **0.29** | **0.15** | **0.31** | **0.22** | | | |
| Rt | 3.71 | 1.79 | 1.44 | 0.17 | 0.13 | 0.82 | 0.85 | 0.35 | 3.00 | | |
| | **1.17** | **0.53** | **0.47** | **0.31** | **0.33** | **0.24** | **0.47** | **0.31** | **0.42** | | |
| Bk | 3.83 | 1.30 | 2.04 | 0.33 | 0.55 | 0.71 | 0.59 | 0.56 | 0.41 | 4.72 | |
| | **1.31** | **0.72** | **0.59** | **0.28** | **0.44** | **0.26** | **0.58** | **0.33** | **0.68** | **0.42** | |
| Ot | 3.35 | 1.49 | 1.87 | 0.77 | 0.28 | 0.82 | 0.86 | 0.02 | 0.13 | 0.68 | 4.91 |
| | **1.40** | **0.71** | **0.44** | **0.20** | **0.41** | **0.19** | **0.43** | **0.20** | **0.53** | **0.25** | **0.31** |
| | D. GMM Specification 5: Estimate of $X \times 100$ with standard errors | | | | | | | | | | |
| Fd | 4.98 | | | | | | | | | | |
| | **0.57** | | | | | | | | | | |
| Cl | 2.51 | 4.77 | | | | | | | | | |
| | **1.33** | **0.59** | | | | | | | | | |
| Bm | 3.74 | 1.19 | 5.10 | | | | | | | | |
| | **1.32** | **0.63** | **0.57** | | | | | | | | |
| Mn | 3.11 | 1.69 | 2.43 | 6.18 | | | | | | | |
| | **1.04** | **0.98** | **0.47** | **0.47** | | | | | | | |
| Oi | 2.75 | 0.41 | 1.51 | 1.98 | 5.16 | | | | | | |
| | **1.05** | **0.40** | **0.32** | **0.26** | **0.38** | | | | | | |
| Hw | 3.07 | 1.04 | 1.61 | 1.17 | 0.45 | 5.89 | | | | | |
| | **1.40** | **0.67** | **0.42** | **0.26** | **0.26** | **0.49** | | | | | |
| Ch | 3.12 | 0.67 | 1.78 | 1.55 | 0.05 | 2.52 | 5.60 | | | | |
| | **1.95** | **0.67** | **0.58** | **0.60** | **0.37** | **0.89** | **0.66** | | | | |
| Bx | 3.23 | 1.07 | 1.37 | 1.05 | 0.48 | 0.81 | 0.53 | 4.31 | | | |
| | **1.15** | **0.41** | **0.29** | **0.16** | **0.20** | **0.21** | **0.27** | **0.33** | | | |
| Rt | 3.72 | 1.54 | 1.26 | 0.39 | 0.05 | 0.84 | 0.63 | 0.50 | 3.96 | | |
| | **1.03** | **0.31** | **0.22** | **0.17** | **0.15** | **0.13** | **0.23** | **0.12** | **0.40** | | |
| Bk | 4.24 | 1.60 | 2.21 | 0.49 | 0.70 | 0.96 | 0.83 | 0.77 | 0.70 | 5.67 | |
| | **1.08** | **0.54** | **0.49** | **0.41** | **0.26** | **0.27** | **0.61** | **0.29** | **0.25** | **0.60** | |
| Ot | 3.75 | 1.63 | 1.89 | 1.01 | 0.40 | 0.94 | 0.92 | 0.23 | 0.41 | 0.70 | 5.65 |
| | **1.24** | **0.52** | **0.28** | **0.24** | **0.24** | **0.24** | **0.31** | **0.17** | **0.14** | **0.26** | **0.40** |

Table 3: **Second moments parameter estimates for GMM Sp2, Sp3, Sp4, and Sp5**
This table presents the GMM estimates of $X$ for the GMM specification 2 (**Sp2**) in Panel A, specification 3 (**Sp3**) in panel B, specification 4 (**Sp4**) in panel C, and specification 5 (**Sp5**) in panel D. This table complements the information in Table 2. GMM specification **Sp1** considers means, variances, covariances, and skewness; **Sp2** adds coskewness to the moments in **Sp1**; **Sp3** adds kurtoses to the moments in **Sp1**; **Sp4** adds kurtoses to the moments in **Sp2**; and **Sp5** considers all moments together.

| Asset | $\mu_i - r_f$ | $\sigma_i$ | $S_i$ | $\delta_i$ | $\frac{\delta_i}{S_i}$ | $w_i^{\mathbf{MV}}$ | $w_i^{\mathbf{AV}}$ |
|---|---|---|---|---|---|---|---|
| Fd | 0.0069 | 0.049 | 0.14 | 0.46 | 3.26 | 0.63 | 0.01 |
| Cl | 0.0066 | 0.063 | 0.11 | 0.62 | 5.89 | 0.13 | 0.61 |
| Bm | 0.0071 | 0.072 | 0.10 | 0.66 | 6.64 | -0.35 | 0.48 |
| Mn | 0.0072 | 0.075 | 0.10 | 0.47 | 4.87 | 0.10 | -0.13 |
| Oi | 0.0067 | 0.065 | 0.10 | 0.57 | 5.53 | 0.11 | 0.38 |
| Hw | 0.0091 | 0.075 | 0.12 | 0.51 | 4.23 | 0.24 | -0.07 |
| Ch | 0.0092 | 0.087 | 0.11 | 0.67 | 6.38 | 0.05 | 0.52 |
| Bx | 0.0079 | 0.062 | 0.13 | 0.53 | 4.20 | 0.20 | 0.00 |
| Rt | 0.0078 | 0.061 | 0.13 | 0.46 | 3.60 | 0.15 | -0.57 |
| Bk | 0.0085 | 0.074 | 0.12 | 0.42 | 3.66 | 0.18 | -0.16 |
| Ot | 0.0041 | 0.075 | 0.05 | 0.48 | 8.92 | -0.44 | -0.07 |
| **MV** | 0.0093 | 0.053 | 0.18 | 0.44 | 2.48 | | |
| **AV** | 0.0072 | 0.075 | 0.10 | 0.81 | 8.40 | | |

Table 4: **Characteristics of Individual Industry Portfolios and Mutual Funds**
This table presents the risk premium ($\mu - r_f$), volatility ($\sigma_i$), Sharpe ratio ($S_i$), non-normality parameter ($\delta_i$) and the non-normality ratio ($\frac{\delta_i}{S_i}$) of the industry portfolios, the mean-variance fund (MV), and the non-normality-variance fund (AV). The last two columns present, respectively, the weights of the industry portfolios in the MV fund and the AV fund.

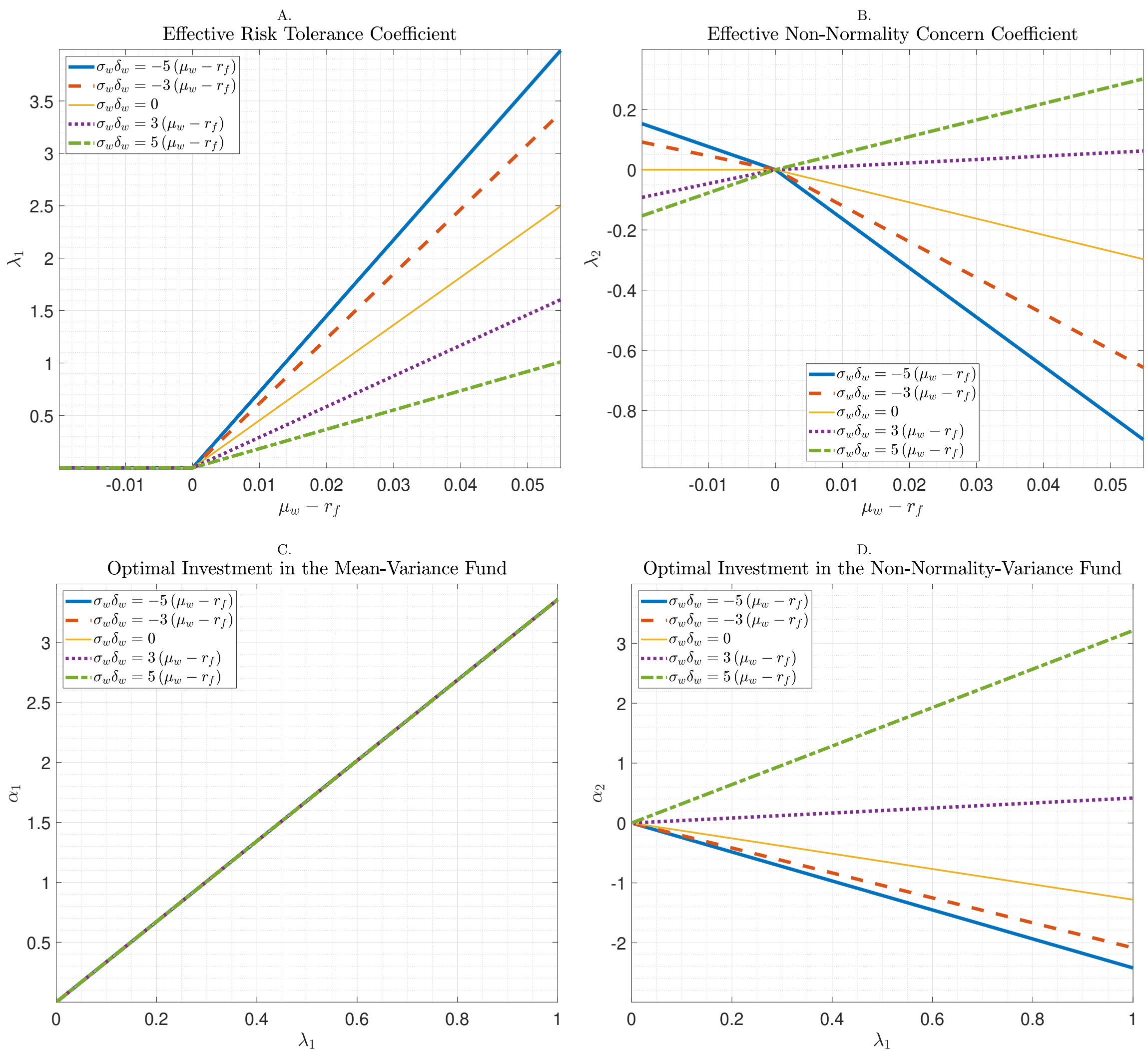


Figure 1: **Endogenous Preference Parameters and Optimal Certainty Equivalent**
Panels A and B present the evolution of investor's (endogenous) effective risk tolerance coefficient ($\lambda_1$) and investor's effective non-normality concern coefficient ($\lambda_2$) as a function of the targeted expected excess return ($\mu_w - r_f$) for different investors' targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$). Panels C and D present the evolution of the investor's optimal positions in the mean-variance fund ($\alpha_1$) and the non-normality-variance fund ($\alpha_2$) as a function of the effective risk tolerance coefficient ($\lambda_1$) for different investors' targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$).

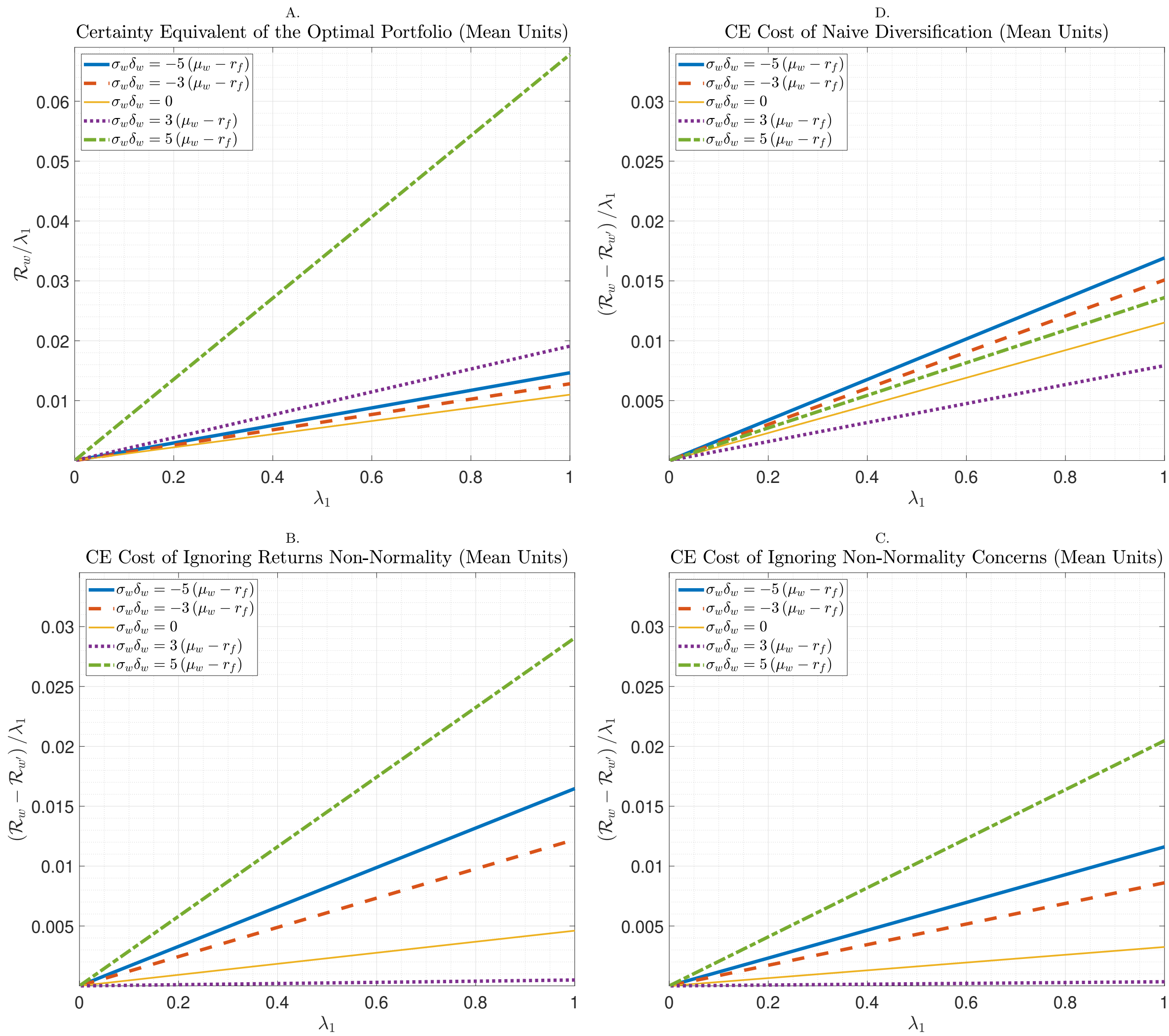


Figure 2: **Certainty Equivalent Costs of Suboptimal Portfolios per risk tolerance**

Panel A presents the evolution of the investor's certainty equivalent expressed in mean units ($\frac{\mathcal{R}_w}{\lambda_1}$) as a function of effective risk tolerance coefficient ($\lambda_1$) for different investor's targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$). Panels B, C and D present the certainty equivalent (CE) cost (in mean units) of ignoring returns non-normality, the certainty equivalent (CE) cost (in mean units) of ignoring non-normality concerns, and the certainty equivalent (CE) cost (in mean units) of naive diversification, respectively, as functions of the effective risk tolerance coefficient ($\lambda_1$) for different investor's targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$). In Panel B, the investor behaves as if returns were normally distributed when solving her optimization problem. In Panel C, (s)he ignores the non-normality constraint in her portfolio optimization problem and behaves as a mean-variance optimizer. In Panel D, (s)he chooses the equally weighted portfolio instead of the optimal portfolio.

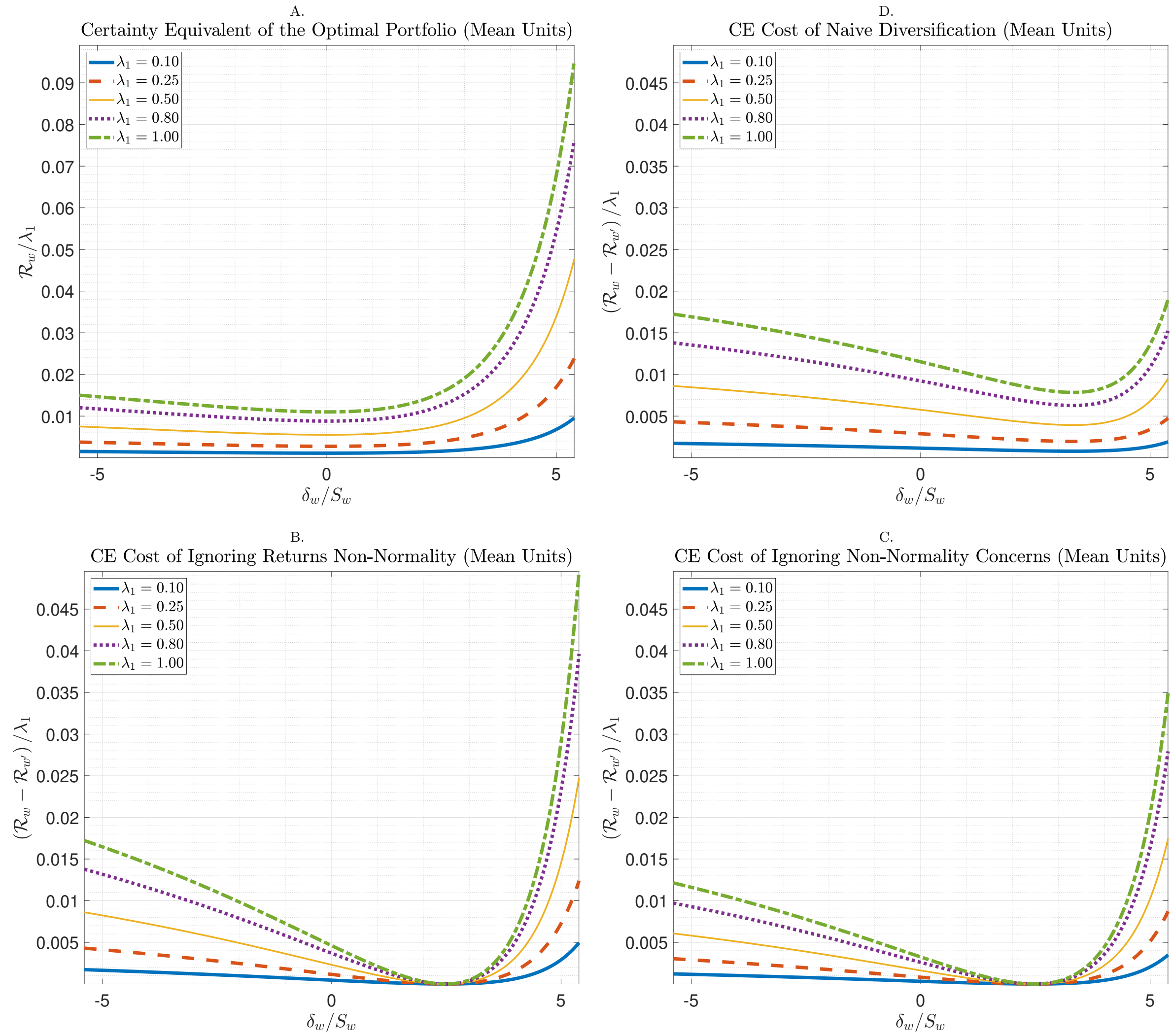


Figure 3: **Certainty Equivalent Costs of Suboptimal Portfolios per non-normality ratio** Panel A presents the evolution of the investor's certainty equivalent expressed in mean units ($\frac{\mathcal{R}_w}{\lambda_1}$) as a function of the non-normality ratio $\frac{\delta_w}{S_w}$, for different investors' effective risk tolerance coefficient ($\lambda_1$). Panels B, C, and D present respectively the certainty equivalent (CE) cost (in mean units) of ignoring returns non-normality, the certainty equivalent (CE) cost (in mean units) of ignoring non-normality concerns and the certainty equivalent (CE) cost (in mean units) of naive diversification as a function of the non-normality ratio $\frac{\delta_w}{S_w}$ for different investor's effective risk tolerance coefficient ($\lambda_1$). In Panel B, investor behaves as if returns were normally distributed when solving her optimization problem. In Panel C, (s)he ignores the non-normality constraint in her portfolio optimization problem and behaves as a mean-variance optimizer. In Panel D, (s)he chooses the equally weighted portfolio instead of the optimal portfolio.

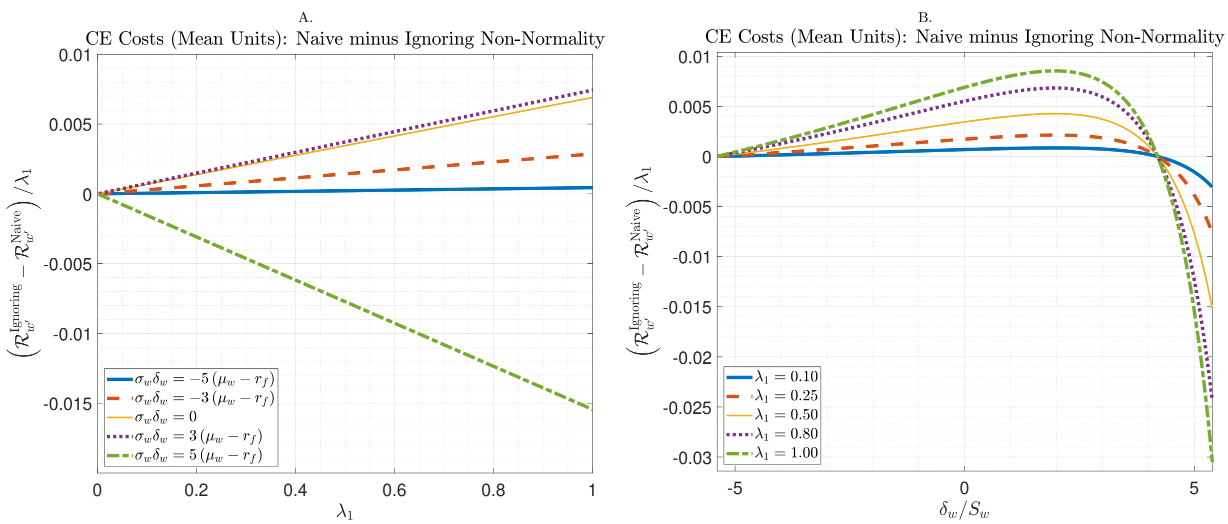


Figure 4: **Comparing Certainty Equivalent Costs of Suboptimal Portfolios**

Panel A presents the evolution of the difference in the investor's certainty equivalents of ignoring non-normality and of naive diversification expressed in mean units, $\left(\mathcal{R}_{w'}^{\text{Ignoring}} - \mathcal{R}_{w'}^{\text{Naive}}\right)/\lambda_1$ as a function of the effective risk tolerance coefficient ($\lambda_1$) for different investors' targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$). Panel B presents the evolution of the difference in investors' certainty equivalents of ignoring non-normality and of naive diversification expressed in mean units, $\left(\mathcal{R}_{w'}^{\text{Ignoring}} - \mathcal{R}_{w'}^{\text{Naive}}\right)/\lambda_1$ as a function of the investor's targeted portfolio non-normality ratio ($\frac{\delta_w}{S_w}$) for different effective risk tolerance coefficient ($\lambda_1$).

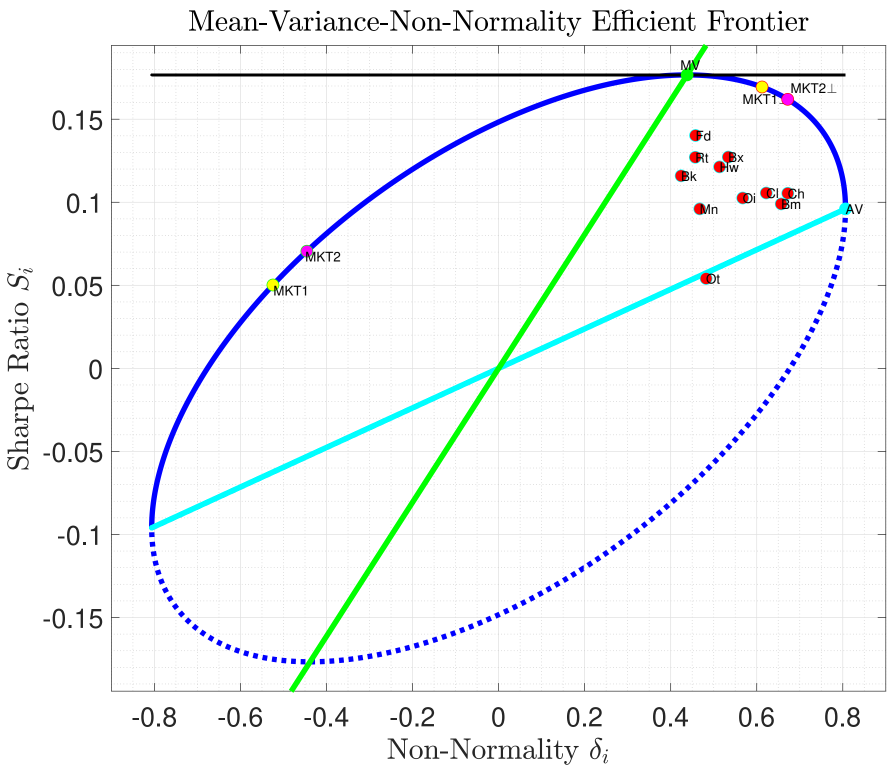


Figure 5: **Efficient frontier**

Mean-variance-non-normality efficient frontier that results from solving the investor optimization problem with the industry portfolios asset menu. The efficient frontier is the bold (upper) part of the ellipse that characterizes the investor's optimal portfolio combination of degree of non-normality and Sharpe ratio. **MV** and **AV** represent, respectively, the mean-variance fund and the non-normality-variance fund. MKT1 and MKT2 are, respectively, the value-weighted market portfolio and the constant-average value-weighted market portfolio while MKT1 $\perp$ and MKT2 $\perp$ are their orthogonal counterparts, respectively.

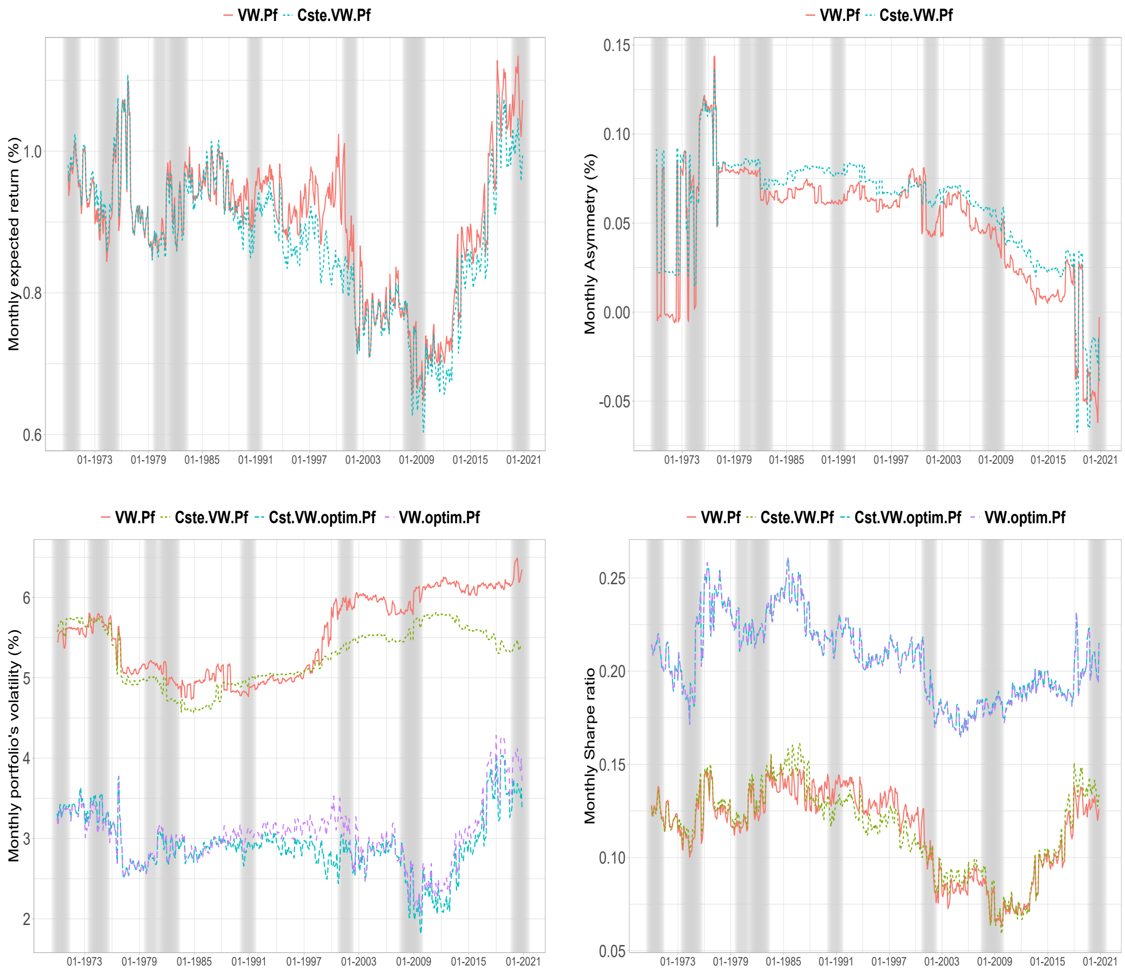


Figure 6: **Expected return, volatility, non-normality, and Sharpe ratio dynamics**
Upper left panel represents the time evolution of the monthly average return on the value-weighted market portfolio (VW.Pf) and the constant-average value-weighted market portfolio (Cste.VW.Pf). The upper right panel represents the time evolution of the monthly non-normality coefficient ($\delta_w$) of the value-weighted market portfolio (VW.Pf) and the constant-average value-weighted market portfolio (Cste.VW.Pf). The bottom left panel represents the time series evolution of the monthly volatility of the value-weighted market portfolio (VW.Pf), the constant-average value-weighted market portfolio (Cste.VW.Pf), and their optimal counterparts which target the same levels of expected return and degree of non-normality (VW.optim.Pf and Cste.VW.optim.Pf, respectively). The bottom right panel represents the time series evolution of the monthly Sharpe ratio of the value-weighted market portfolio (VW.Pf), the constant-average value-weighted market portfolio (Cste.VW.Pf), and their optimal counterparts, which target the same levels of expected return and degree of non-normality (VW.optim.Pf and Cste.VW.optim.Pf, respectively).

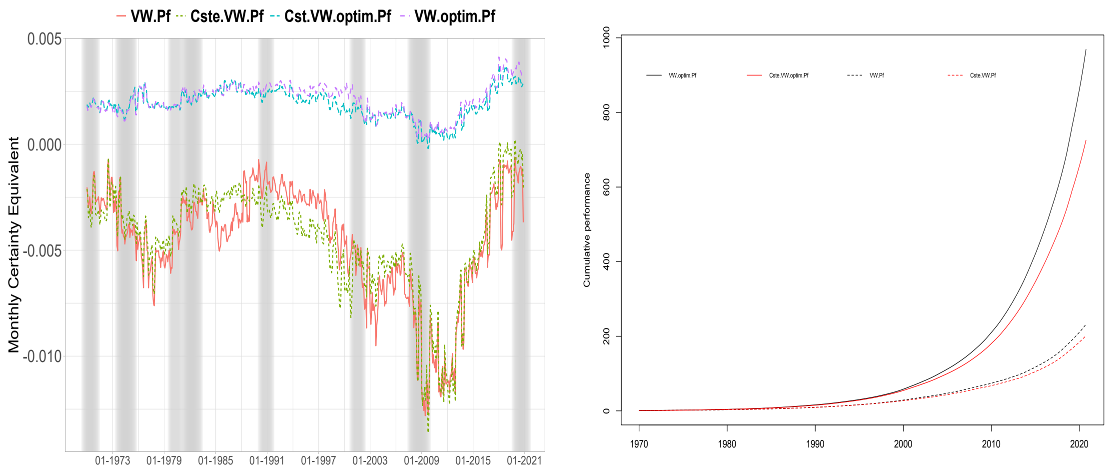


Figure 7: **Welfare dynamics and cumulative performance of the portfolio strategies**
Left panel represents the time series evolution of the monthly certainty equivalent of the value-weighted market portfolio (VW.Pf), the constant-average value-weighted market portfolio (Cste.VW.Pf), and their optimal counterparts, which target the same levels of expected return and degree of non-normality (VW.optim.Pf and Cste.VW.optim.Pf, respectively). The right panel represents cumulative returns of the value-weighted market portfolio (VW.Pf), the constant-average value-weighted market portfolio (Cste.VW.Pf), and their optimal counterparts.

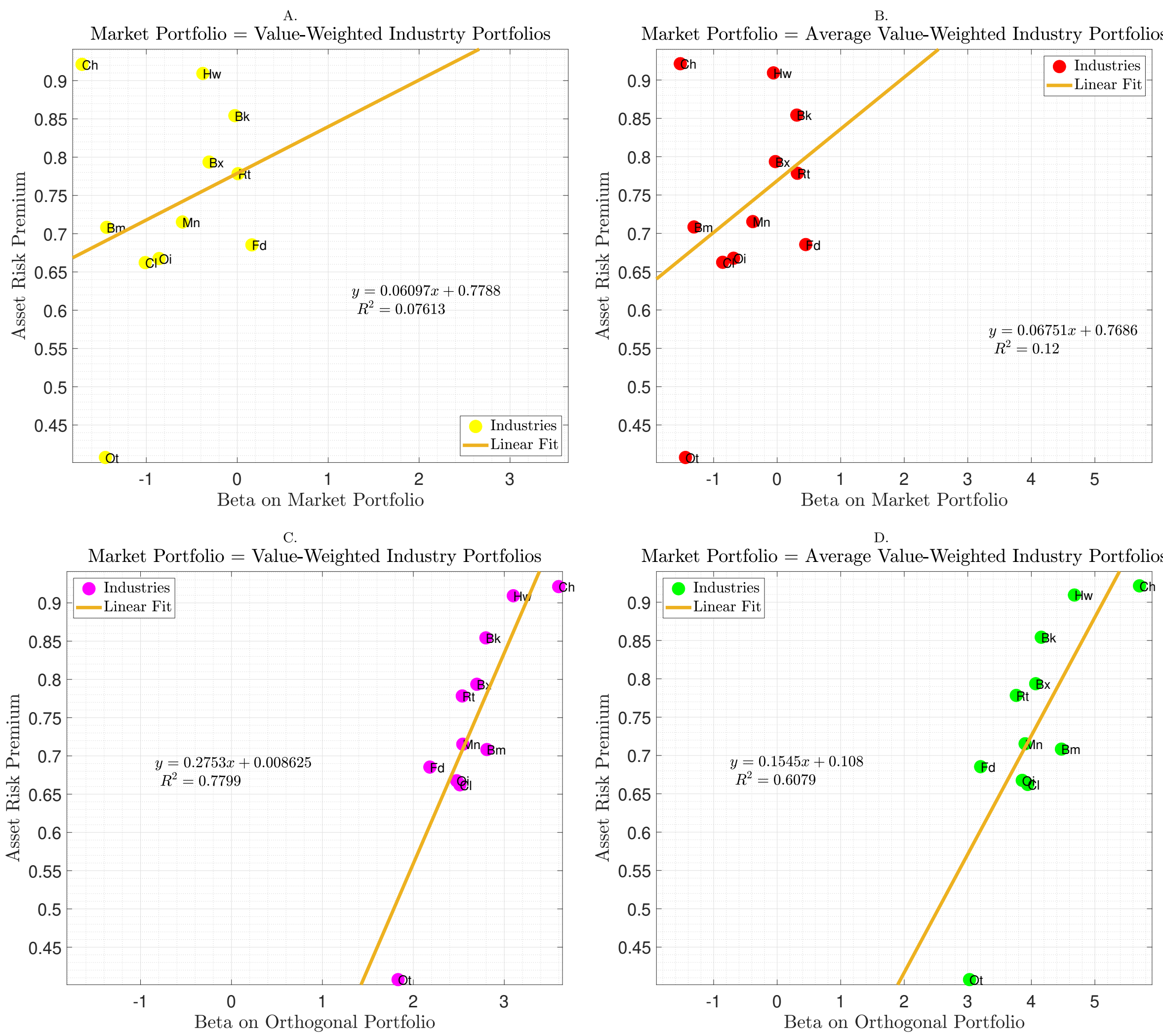


Figure 8: **Asset risk premiums versus asset betas**
Panels A and B represent, respectively, the industry portfolios market risk exposure versus their average risk premium for the value-weighted market portfolio (MKT1) and the constant-average value-weighted market portfolio (MKT2). Panels C and D represent, respectively, the industry portfolios market risk exposure versus their average risk premium for the value-weighted market orthogonal portfolio (MKT1 $\perp$) and the constant-average value-weighted market orthogonal portfolio (MKT2 $\perp$).