Liability-Driven Investment by Disappointment-Averse Managers

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Abstract

We introduce a liability-driven investment framework where the portfolio manager has generalized disappointment aversion preferences. We show that disappointment aversion can be interpreted as a penalty in the manager's objective based on the expected payoff of a put option on the funding ratio return. The optimal strategy of the manager is to allocate the assets of the fund to two risky portfolios, the standard mean-variance efficient portfolio and a liability-hedge portfolio connected to the covariance between assets and liabilities. We analyze how the weights to these two risky portfolios depend on the risk aversion, disappointment aversion, and disappointment threshold of the manager.

Keywords: Downside risk, funding ratio, generalized disappointment aversion, surplus.
JEL Classification: G11, G23.

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1 Introduction

In this paper we focus on the investment decision of a pension fund manager. The portfolio allocation decision of a pension fund manager differs from a standard portfolio choice problem in several key aspects. Arguably the most important aspect is that the manager is restricted by the pension liabilities connected to the fund. She has to take into account the risk generated by the uncertain future value of pension liabilities when making the asset allocation decision. Studies focusing on this liability driven investment (LDI) approach of pension funds constitute a growing area within the portfolio choice literature (e.g., Sharpe and Tint, 1990; Rudolf and Ziemba, 2004; van Binsbergen et al., 2008; Hoevenaars et al., 2008; Berkelaar and Kouwenberg, 2010; Ang et al., 2013; and van Binsbergen and Brandt, 2014).

Another important aspect that has to be considered when modeling the investment decision of a pension fund manager is the significant penalties if the assets of the fund do not meet the liabilities. For example, in the US the 2006 Pension Protection Act requires that a pension plan's funding should equal 100% of the plan's liabilities. Sponsors of severely underfunded plans are required to fund their plans according to special rules that result in higher contributions to the plan. This second aspect is even more important after the recent financial crisis as a lot of corporate pension plans in the US have become underfunded.[1] Thus, a particular focus on avoiding disappointing losses on the asset side seems crucial.

We build and analyze a theoretical model that explicitly takes into account these two aspects of the manager's decision. Our starting point is an LDI framework similar to that of Hoevenaars et al. (2008) and van Binsbergen et al. (2008), who extend the standard portfolio choice problem with power utility by considering a manager who has to take into account a predetermined liability. They show that the focus on liabilities gives rise to a hedging demand in the optimal asset allocation of the manager. The manager will invest some of the

[1] van Binsbergen and Brandt (2014) report that by the end of 2008 the aggregate funding status of US corporate defined benefit retirement plans had fallen to 75% of total liabilities and that more than 50% of private defined benefit plans were less than 80% funded.

fund's assets in a liability-hedge portfolio that best captures the variation in liabilities.

Contrary to the model with standard power preferences over the funding ratio, our manager has a particular focus on downside risk. We model this by assuming that the manager has generalized disappointment preferences as in Routledge and Zin (2010). One desirable feature of these preferences is that they nest the power utility as a special case. By considering a manager with an extra focus on downside risk, we relate to the work of Ang et al. (2013), who present an LDI approach where the fund manager's objective includes a put option penalty for not meeting the liabilities.[2] We argue that disappointment aversion can also be seen as penalty in the manager's objective in the form of a put option. The difference between our setup and that of Ang et al. (2013) is that they introduce the manager's objective in an ad hoc way, while we rely on well-grounded preferences.

We derive an analytical solution to the proposed portfolio choice problem. We show that the optimal portfolio can be decomposed into two parts: a speculative portfolio that corresponds to the standard mean-variance investment strategy and a liability-hedge portfolio that arises due to the presence of liabilities. This is similar to the result of Hoevenaars et al. (2008) and van Binsbergen et al. (2008). Disappointment aversion leads to a change in the relative weights of these two portfolios. Managers with higher degree of disappointment aversion decrease their investment in the mean-variance portfolio and increase their position in the liability-hedge portfolio.

We argue that the disappointment threshold has an important effect on the optimal asset allocation. It generates non-linear patterns in the manager's risk taking. Managers with a low disappointment threshold take on more risk and tilt their asset allocation towards the mean-variance portfolio as there is a low probability of disappointment. Managers with a high disappointment threshold take on more risk in order to increase the probability of avoiding disappointment. Managers with intermediate disappointment thresholds are the

[2]van Binsbergen and Brandt (2014) consider a similar penalty in the form of a disutility for requesting and receiving extra contributions from the plan sponsor.

most conservative in terms of risk taking.

We proceed as follows: Section 2 introduces the theoretical setup. Section 3 discusses the optimal asset allocation strategy of the manager and provides a calibration assessment of the results. Section 4 concludes.

2 Theoretical setup

We consider a manager at time 0 who is concerned about the funding ratio of the pension fund. The funding ratio, F_t, is defined as the total value of the fund's assets, A_t, over the present value of all future liabilities, L_t.[3] The funding ratio at the end of the investment horizon $t = T$ can be written

$$F_T \equiv \frac{A_T}{L_T} = \frac{A_0 R_{A,T}}{L_0 R_{L,T}} = F_0 R_{F,T} \ , \tag{1}$$

where A_0, L_0, and F_0 denote the initial ($t = 0$) values of the assets, liabilities, and the funding ratio, respectively. $R_{A,T}$ and $R_{L,T}$ are the gross returns on assets and liabilities, respectively, and

$$R_{F,T} \equiv \frac{R_{A,T}}{R_{L,T}} \tag{2}$$

is the funding ratio gross return over the investment horizon.

The manager can allocate wealth between N risky assets $i = 1, 2, \ldots, N$ and the risk-free rate $i = 0$. We assume that the risky assets and the liabilities are jointly log-normal:

$$\begin{pmatrix} r_T \\ r_{L,T} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu_L \end{pmatrix}, \begin{pmatrix} \Sigma & \vartheta_L \\ \vartheta_L^\top & \sigma_L^2 \end{pmatrix} \right) \ , \tag{3}$$

[3] Other studies including Sharpe and Tint (1990) and Ang et al. (2013) focus on the fund surplus, defined as $S_t \equiv A_t - L_t$. Note, that the two approaches are closely related as the funding ratio is equal to the fund surplus scaled by the value of the liabilities: $F_t = S_t/L_t + 1$. The advantage of F_t is that it is non-negative by definition and therefore more suited for commonly used power preferences.

where $r_T = (\ln R_{1,T}, ..., \ln R_{N,T})^\top$ and $r_{L,T} = \ln R_{L,T}$. Using the approximation of Campbell and Viceira (2002) for the log return on assets, the log return on the funding ratio is given by

$$r_{F,T} = r_{A,T} - r_{L,T} = w^\top \left(r_T - r_0 \iota + \frac{1}{2}\sigma^2 \right) - \frac{1}{2} w^\top \Sigma w - (r_{L,T} - r_0) \ , \tag{4}$$

where w is the vector of portfolio weights on risky assets, ι is a vector of ones, and σ^2 is the diagonal of Σ. The mean and variance of $r_{F,T}$ is then given by

$$\begin{aligned} \mu_F &= w^\top \left(\mu - r_0 \iota + \frac{1}{2}\sigma^2 \right) - \frac{1}{2} w^\top \Sigma w - (\mu_L - r_0) \\ \sigma_F^2 &= w^\top \Sigma w - 2 w^\top \vartheta_L + \sigma_L^2 \ . \end{aligned} \tag{5}$$

We assume that the manager has generalized disappointment aversion (GDA) preferences. Gul (1991) introduced the original version of disappointment aversion and Routledge and Zin (2010) generalized these preferences. In the original version of the preferences the investor becomes disappointed if the outcome is lower than the certainty equivalent of her investment. Generalized disappointment aversion is a one-parameter extension of Gul's (1991) preferences that allow the disappointment threshold to be different from the certainty equivalent. The fund manager's objective at the initial date $t = 0$ is to maximize the certainty equivalent of the funding ratio at the end of the planning horizon $t = T$. The certainty equivalent of the funding ratio, $\mathcal{R}(F_T)$, is implicitly defined by

$$\theta U(\mathcal{R}(F_T)) = E[U(F_T)] - \ell E[(U(\kappa \mathcal{R}(F_T)) - U(F_T)) I(F_T < \kappa \mathcal{R}(F_T))] \ , \tag{6}$$

where $I(\cdot)$ is an indicator function that takes the value 1 if the condition is met and 0 otherwise, and where

$$U(X) = \begin{cases} \frac{X^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \ln X & \text{if } \gamma = 1 \ . \end{cases} \tag{7}$$

The parameter γ measures the manager's risk aversion. Parameters $\ell \geq 0$ and $\kappa > 0$ determine the degree of disappointment aversion and the threshold of disappointment, respectively. The parameter θ is defined as $1 - \ell \left(\kappa^{1-\gamma} - 1\right) I\left(\kappa > 1\right)$.

When ℓ is equal to zero, θ is equal to one and the second term in the right hand side of (6) disappears. The manager then displays expected utility (EU) preferences. When $\ell > 0$, outcomes lower than $\kappa \mathcal{R}\left(F_T\right)$ receive an extra weight in (6), decreasing the manager's certainty equivalent relative to expected utility. Consequently, a disappointment-averse manager would like to avoid outcomes below $\kappa \mathcal{R}\left(F_T\right)$. The penalty for disappointing outcomes increases with ℓ, which can be interpreted as the degree of disappointment aversion. If $\kappa < 1$, the random future value is considered disappointing if it lies sufficiently below today's certainty equivalent. Alternatively, if $\kappa > 1$, the random future value must be sufficiently far above the manager's certainty equivalent to be considered not disappointing.

The manager's objective of maximizing $\mathcal{R}\left(F_T\right)$ is equivalent to maximizing its logarithm, $\eta \equiv \ln \mathcal{R}\left(F_T\right)$. In the Appendix we show that η is implicitly defined by

$$\eta = \mu_F - \frac{\gamma - 1}{2}\sigma_F^2 + \frac{1}{\gamma - 1}\ln\left(\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left(\left(\gamma - 1\right) p_{F,T}\right)\right] - 1\right)\right) \ , \tag{8}$$

where

$$p_{F,T} \equiv \max\left(\ln\kappa + \eta - r_{F,T}, 0\right) \ . \tag{9}$$

To gain more intuition about the manager's objective, consider first the case with no disappointment aversion. Setting $\ell = 0$, substituting in (5) into (8), and dropping the constant terms that do not change the maximization problem, the manager's objective becomes

$$\max_{w} \quad w^\top \left(\mu - r_0 \iota + \frac{1}{2}\sigma^2\right) - \frac{\gamma}{2} w^\top \Sigma w + \left(\gamma - 1\right) w^\top \vartheta_L \ . \tag{10}$$

The first two terms in (10) correspond to a standard mean-variance objective concerning only the asset side of the fund. The last term captures the effect of the liabilities. The manager takes into account the covariation between liabilities and asset returns.

According to (8), the disappointment-averse manager has an extra term in her objective, which is a non-linear function of $p_{F,T}$. As it can be seen in (9), $p_{F,T}$ is the payoff of a European put option on the funding ratio return $r_{F,T}$, with a strike equal to the manager's endogenous threshold of disappointment, $\ln \kappa + \eta$. If the funding ratio return falls below the disappointment threshold, the put option matures in the money. This additional term lowers the certainty equivalent of the manager.[4] Therefore, disappointment aversion can be interpreted as an endogenous penalty for downside risks in the manager's certainty equivalent. The penalty is based on the expected payoff of this option.

To see this more clearly, consider the case $\kappa \leq 1$ and use $\exp(x) \approx 1+x$ and $\ln(x) \approx x-1$.[5] Using these approximations, the certainty equivalent becomes

$$\eta \approx \mu_F - \frac{\gamma - 1}{2}\sigma_F^2 - \ell\kappa^{1-\gamma} E\left[p_{F,T}\right] \ . \tag{11}$$

Then the downside risk penalty is linear in the expected payoff of the put option and the importance of the penalty term is mainly determined by the manager's degree of disappointment aversion ℓ. The log certainty equivalent in (8) contains the expectation of a nonlinear function of the put option payoff and the penalty term is also a nonlinear function of this expectation, but the effect is similar as in (11). Note that when $U(\cdot)$ is the logarithmic utility (i.e., when $\gamma \to 1$) equation (11) is not an approximation, but holds as an exact equality.

[4]The term within $\ln(\cdot)$ is smaller than one, so the last term in (8) is negative. Also note that the coefficient θ ensures that the term within $\ln(\cdot)$ is positive.

[5]When $\kappa \leq 1$, the disappointment threshold is low, so the value of $p_{F,T}$ is zero or close to zero for most of the outcomes. Hence, the approximation $\exp(x) \approx 1 + x$ works well. In turn, the value of $E\left[\exp\left((\gamma - 1)\, p_{F,T}\right)\right]$ is around one. Also considering that $\theta = 1$ when $\kappa \leq 1$, the value of $\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left((\gamma - 1)\, p_{F,T}\right)\right] - 1\right)$ will be close to one, so $\ln(x) \approx x - 1$ is also a good approximation. For cases with $\kappa > 1$, these approximations can be rather inaccurate.

3 Results and calibration

3.1 The optimal asset allocation

Given our distributional assumptions on the asset returns, we show in the Appendix that the expectation in the downside risk penalty part of the certainty equivalent (8) is

$$E\left[\exp\left(\left(\gamma-1\right)p_{F,T}\right)\right]-1=S\Phi\left(d_2\right)-\Phi\left(d_1\right)\ , \tag{12}$$

where

$$S\equiv\exp\left(\left(\gamma-1\right)\left(\ln\kappa+\eta-\mu_F\right)+\frac{1}{2}\left(\gamma-1\right)^2\sigma_F^2\right)\ , \tag{13}$$

$$d_1\equiv\frac{\ln\kappa+\eta-\mu_F}{\sigma_F}\qquad\text{and}\qquad d_2\equiv d_1+\left(\gamma-1\right)\sigma_F. \tag{14}$$

The key determinants of (12) are the disappointment threshold $\ln\kappa+\eta$ and the mean and variance of the funding ratio return. We study how these variables effect the downside risk penalty in the calibration section of the paper.

It is shown in the Appendix that the optimal asset allocation for the pension fund manager is

$$w=\frac{1}{\tilde{\gamma}}w^{\mathbf{MV}}+\left(1-\frac{1}{\tilde{\gamma}}\right)w^{\mathbf{LH}}\ . \tag{15}$$

The fund manager invests in two risky portfolios that are defined as

$$w^{\mathbf{MV}}\equiv\Sigma^{-1}\left(\mu-r_0\iota+\frac{1}{2}\sigma^2\right)\qquad\text{and}\qquad w^{\mathbf{LH}}\equiv\Sigma^{-1}\vartheta_L\ . \tag{16}$$

Portfolio $w^{\mathbf{MV}}$ is the solution to the standard mean-variance problem on the asset side of the fund and we refer to it as the mean-variance portfolio. The portfolio $w^{\mathbf{LH}}$ is based on the covariance between liabilities and assets. Assets that are positively correlated with the return on liabilities have a positive weight in $w^{\mathbf{LH}}$, while negatively correlated assets have a negative weight. That is, by investing in this portfolio the manager can lower the variance of the funding ratio. In fact, it can been seen from (5) that $w^{\mathbf{LH}}$ minimizes σ_F^2. We refer to $w^{\mathbf{LH}}$ as the liability-hedge portfolio.

The weight attached to these portfolios depends on the coefficient $\tilde{\gamma}$, for which the analytical formula is given by

$$\tilde{\gamma} = \gamma + \frac{1}{\sigma_F} \cdot \frac{\ell \kappa^{1-\gamma} S \phi\left(d_2\right)}{\theta + \ell \kappa^{1-\gamma} \Phi\left(d_1\right)} \ . \tag{17}$$

We refer to $\tilde{\gamma}$ as the effective risk aversion of the manager. The inverse of $\tilde{\gamma}$ equals the weight that the manager assigns to the mean-variance portfolio. When $\tilde{\gamma} = 1$, the optimal asset allocation is identical to the mean-variance portfolio. This manager does not care about the liabilities connected to the pension plan. As her effective risk aversion increases, the fund manager decreases the position taken in $w^{\mathbf{MV}}$, and increases her position in the liability-hedge portfolio $w^{\mathbf{LH}}$. As $\tilde{\gamma} \to \infty$, the optimal asset allocation of the manager becomes identical to $w^{\mathbf{LH}}$.

The effective risk aversion completely characterizes the optimal asset allocation of the fund manager in our setup. The main question is how $\tilde{\gamma}$ changes with different preference parameter values. When the fund manager does not care about disappointment, her effective risk aversion is simply the risk aversion parameter of the power utility. This scenario corresponds to the setup considered by Hoevenaars et al. (2008) and van Binsbergen et al. (2008). However, when the manager is disappointment averse, $\tilde{\gamma} > \gamma$ holds. That is, disappointment aversion leads to higher effective risk aversion.

In order to find the optimal asset allocation, equations (8) and (15) have to be solved simultaneously. Thus, the downside risk penalty and the effective risk aversion are endogenous under disappointment aversion preferences. To see how these quantities behave in a specific example, we turn to a calibration exercise.

3.2 Calibration assesment

The calibration involves a risk-free asset labelled as cash and two risky assets: a stock and a bond. Our setup easily handles multiple asset classes; however, two risky assets in the portfolio are enough to convey the main results. Considering additional asset classes changes the composition of the mean-variance and liability-hedge portfolios, but the intuition behind the choice of the fund manager remains the same as for two risky assets.

We use the same calibration as Ang et al. (2013). The S&P 500 total return index represents equity and the Ibbotson US long-term corporate bond total return index represents the bond. The sample period is between January 1952 and December 2011. We assume that the liability has the same expected return as the bond and a slightly higher volatility. As Ang et al. (2013) and van Binsbergen and Brandt (2014) argue, long-term bonds represent an asset class that are highly correlated with liabilities. Therefore, the correlation between them is set to 0.98. Our assumptions on the return distributions are summarized in Table 1. The investment horizon of the manager is assumed to be one year.

Given these assumptions, the composition of the mean-variance and liability-hedge portfolios are

$$w^{\mathbf{MV}} = \begin{pmatrix} 331.9\% \\ 303.1\% \end{pmatrix} \qquad \text{and} \qquad w^{\mathbf{LH}} = \begin{pmatrix} 7.6\% \\ 110.7\% \end{pmatrix} . \tag{18}$$

In both of the above portfolios, the first entry corresponds to the stock and the second entry to the bond. The mean-variance portfolio assigns a positive weight to the risky assets as both of them has a positive expected excess return. The weight on the stock is higher, because it has a higher Sharpe ratio than the bond. The liability-hedge portfolio also assigns a positive weight to both assets, but the relative weight assigned to the bond is much larger than that of the stock. Because of their high correlation, the bond provides a much better hedge against changes in the value of the liabilities.

We begin by considering the choice of an expected utility manager with $\gamma = 5$, who does not take into account liabilities. Her optimal asset allocation is

$$w = \frac{1}{\gamma} w^{\mathbf{MV}} = \begin{pmatrix} 66.4\% \\ 60.6\% \end{pmatrix} . \quad (19)$$

She invests around 60% in both risky assets and takes a short position in the risk-free asset. If she also takes into account the liabilities connected to the plan, her optimal asset allocation is described by (15) with $\tilde{\gamma} = 5$. That leads to 72.5% investment in the stock and 149.2% in the bond. A liability driven fund manager chooses a considerably larger position in the bond because of its high positive correlation with the liabilities.

What happens if the fund manager is disappointment averse? To see how disappointment aversion changes the optimal asset allocation, we consider mangers with $\gamma = 5$, $\kappa = 1$ (we study the effect of κ later), and different values of ℓ. We restrict our attention to portfolios of the form (15) and study what happens when the fund manager varies the weight attached to the mean-variance portfolio. Note that the optimal asset allocation is where the weight on the mean-variance portfolio is $1/\tilde{\gamma}$.

Panel A of Figure 1 shows how the log certainty equivalent η of the manager changes with the weight on the mean-variance portfolio. Recall that the objective of the manager is to maximize η. On each line in Panel A, the marker corresponds to the optimal portfolio, where η is maximized. The solid line corresponds to the case with no disappointment aversion. When $\ell = 0$, there is no disappointment penalty in the manager's certainty equivalent (i.e., the last term in (8) is zero). The optimal asset allocation is the one with 0.2 weight on $w^{\mathbf{MV}}$, as it is also implied by the fact that $\tilde{\gamma} = \gamma$ when $\ell = 0$.

For a disappointment-averse manager, there is an endogenous penalty for downside risks. Panel B shows the expectation $E\left[\exp\left(\left(\gamma - 1\right) p_{F,T}\right)\right] - 1$, where $p_{F,T}$ is the payoff of a put option on the funding ratio return with the strike $\ln \kappa + \eta$. In the case $\kappa = 1$, the strike price of the option is the certainty equivalent (displayed in Panel A). There are two observations to highlight from Panel B. First, as the investor increases the weight assigned to the mean-variance portfolio, the variance of the funding ratio return σ_F increases, so the expectation connected to the option payoff also increases.[6] Second, for a given weight in $w^{\mathbf{MV}}$ (i.e., a given level of μ_F and σ_F) a higher strike price (corresponding to a lower ℓ) leads to a higher expected value. Therefore, Panel B of Figure 1 shows that $E\left[\exp\left((\gamma-1)\,p_{F,T}\right)\right]-1$ is similar to the expected payoff of a put option on the funding ratio return.

The downside risk penalty is a nonlinear function of this expectation and it is presented in Panel C. Higher degree of disappointment aversion leads to a larger penalty. According to (8), the certainty equivalent of the fund manager is the sum of the mean-variance certainty equivalent and the downside risk penalty. That is, lines in Panel A for values $\ell > 0$ equal the solid line from the same graph plus the corresponding lines from Panel C. Panel A shows that managers with higher degree of disappointment aversion choose a smaller weight in the mean-variance portfolio. In other words, as ℓ increases, the effective risk aversion $\tilde{\gamma}$ of the manager that determines the optimal asset allocation through (15) also increases.

Panel A of Figure 2 presents how the optimal weight in the mean-variance portfolio, or equivalently, the inverse of the effective risk aversion $1/\tilde{\gamma}$ changes with the manager's disappointment aversion. Panel B presents the corresponding risky asset weights. The manager decreases her position in both risky assets, but the decrease in the stock weight is relatively larger. As her effective risk aversion increases, the manager not only decreases her position in $w^{\mathbf{MV}}$ but also increases her position in $w^{\mathbf{LH}}$ in order to achieve a lower variance of the funding ratio return. Since the liability-hedge portfolio assigns a large weight to the bond, the overall decrease in the bond position is relatively lower.

[6] Recall, that the liability-hedge portfolio minimizes the variance of the funding ratio return. As the manager increases the weight attached to the mean-variance portfolio (and at the same time decreases the weight attached to $w^{\mathbf{LH}}$), σ_F will increase.

3.2.1 The effect of the disappointment threshold on the optimal asset allocation

The initial funding ratio has a large impact on the optimal asset allocation of the manager in Ang et al. (2013). They argue that the manager chooses the least risky portfolios when A_0/L_0 is around one. On the one hand, when the funding ratio is low (A_0/L_0 is lower than one), the manager takes on more risk to bet on the chance that liabilities can be covered at the end of the period. On the other hand, when the funding ratio is very high (A_0/L_0 is well above one), the manager takes on more risk as liabilities are already well covered.

In our setup, the initial funding ratio does not have an effect on the optimal portfolio. This is similar to the result that initial wealth does not have an effect in standard portfolio choice problems with power utility investors. However, changes in κ has a similar effect as the initial funding ratio for Ang et al. (2013). They assume, in an ad hoc way, that the pension fund manager has a downside risk penalty equal to the price of a put option on the funding surplus, with a strike corresponding to $A_T = L_T$. In their setup, the initial funding ratio, A_0/L_0 changes the moneyness of this option. For a disappointment-averse manager, the endogenous penalty for downside risks can be seen as the expected payoff of put option on the funding ratio return with the strike price $\ln \kappa + \eta$. Thus, our setup gives rise to a similar interpretation as that of Ang et al. (2013), but it has the advantage of relying on a more well-grounded preference specification.

In our model κ has a large effect on the strike of the put option and the optimal asset allocation. Panel A of Figure 3 shows how the inverse of the effective risk aversion changes with κ. The highest effective risk aversion and therefore the lowest investment in the mean-variance portfolio corresponds to $\kappa = 1$. When κ decreases from one, the put option's strike becomes lower. Consequently, the expected option payoff and the disappointment penalty in the manager's certainty equivalent decreases. Because of the lower penalty, the manager takes on larger risk by investing more in the mean-variance portfolio. This corresponds to the case when the fund is initially over-funded in the setup of Ang et al. (2013).

When κ increases from one, the disappointment threshold of the manager becomes higher. Consequently, she has to take on more risk in order to avoid disappointment at the end of the period. As it is illustrated in Panel A, the manager invests more in the mean-variance portfolio as κ increases from one. This corresponds to the case when the fund is initially under-funded in the setup of Ang et al. (2013).

Parameter κ changes the disappointment threshold and therefore has a large impact on the manager's risk taking. In this setting, κ can be viewed as a preference parameter that is influenced by the current circumstances of the fund. For example, when the fund is under-funded and the manager has to achieve a high return on the funding ratio, the value of κ will be high. Panel B shows the disappointment threshold, in terms of log return on the funding ratio, that corresponds to different values of κ in our calibration.

4 Conclusion

In this paper we solve the portfolio choice problem of a pension fund manager. Our setup has two important features. First, the manager takes into account the predetermined liability component of the fund. Second, we assume that the manager has generalized disappointment aversion preferences that lead to a special attention on downside risk.

We show that the manager allocates the fund's assets to two portfolios, a speculative portfolio that corresponds to the standard mean-variance optimal investment strategy and a liability-hedge portfolio that arises due to the presence of liabilities. The weights attached to these two portfolios depend on the manager's degree of risk aversion, disappointment aversion, and disappointment threshold. In general, disappointment aversion induces the manager to take on less risk and invest a larger part of the fund in the liability-hedge portfolio. However, different disappointment thresholds lead to different endogenous risk taking.

The setup in this paper can easily be extended by relaxing the assumption of log-normal asset returns. Dahlquist et al. (2014) study the portfolio choice of a disappointment-averse investor when asset log returns follow a skew-normal distribution. This distribution takes into account the individual skewness of assets as well as the coskewness between them. In such an extended setup the manager would allocate funds to a third type of portfolio that is related to the coskewness between liabilities and the securities in the asset side of the fund. It would be interesting to take the implications of the model to asset allocation data of pension funds. Observing actual investment in different asset classes may enable us to identify the preference parameters of the managers. This challenge is left for future research.

A Apppendix

A.1 The log certainty equivalent

We will use the shorthand notation $\mathcal{R}$ to denote $\mathcal{R}(F_T)$ in the appendix. Recall that $r_{F,T} = \ln R_{F,T}$ and $\eta = \ln(\mathcal{R})$. We have

$$\begin{aligned} U(\kappa\mathcal{R}) - U(R_{F,T}) &= U(\kappa\mathcal{R})\left(1 - \frac{U(R_{F,T})}{U(\kappa\mathcal{R})}\right) = U(\kappa\mathcal{R})\left(1 - \left(\frac{R_{F,T}}{\kappa\mathcal{R}}\right)^{1-\gamma}\right) \\ &= U(\kappa\mathcal{R})\left(1 - \exp\left((\gamma-1)\left(\ln\kappa + \eta - r_{F,T}\right)\right)\right). \end{aligned} \tag{A.1}$$

Note that $\forall a, X \in \mathbb{R}$

$$(1 - \exp(aX)) I(X > 0) = 1 - \exp(aXI(X > 0)) = 1 - \exp(a\max(X, 0)) \ . \tag{A.2}$$

This implies

$$\begin{aligned} &E\left[\left(U(\kappa\mathcal{R}) - U(R_{F,T})\right) I(R_{F,T} < \kappa\mathcal{R})\right] \\ &= U(\kappa\mathcal{R}) E\left[\left(1 - \exp\left((\gamma-1)\left(\ln\kappa + \eta - r_{F,T}\right)\right)\right) I(r_{F,T} < \ln\kappa + \eta)\right] \\ &= \kappa^{1-\gamma} U(\mathcal{R})\left(1 - E\left[\exp\left((\gamma-1) p_{F,T}\right)\right]\right), \end{aligned} \tag{A.3}$$

where

$$p_{F,T} \equiv \max\left(\ln\kappa + \eta - r_{F,T}, 0\right).$$

Substituting out equation (A.3) in equation (6) and solving for $U(\mathcal{R})$, we arrive at

$$U(\mathcal{R}) = \frac{E\left[U(R_{F,T})\right]}{\theta + \ell\kappa^{1-\gamma}\left(1 - E\left[\exp\left((\gamma-1) p_{F,T}\right)\right]\right)}. \tag{A.4}$$

Multiplying both sides by $1-\gamma$ sides and taking logs, we get

$$\ln\mathcal{R}^{1-\gamma} = \ln E\left[R_{F,T}^{1-\gamma}\right] - \ln\left(\theta + \ell\kappa^{1-\gamma}\left(1 - E\left[\exp\left((\gamma-1) p_{F,T}\right)\right]\right)\right) \ , \tag{A.5}$$

which leads to

$$\eta = \frac{1}{1-\gamma} \ln E\left[\exp\left((1-\gamma) r_{F,t}\right)\right] + \frac{1}{\gamma-1} \ln\left(\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left((\gamma-1) p_{F,T}\right)\right] - 1\right)\right) \ . \quad \text{(A.6)}$$

Since $r_{F,t} \sim N\left(\mu_F, \sigma_F^2\right)$,

$$\frac{1}{1-\gamma} \ln E\left[\exp\left((1-\gamma) r_{F,t}\right)\right] = \mu_F - \frac{\gamma-1}{2}\sigma_F^2 \ . \quad \text{(A.7)}$$

Substituting this into (A.6) leads to (8) in the main text.

Using the definition of $p_{F,T}$,

$$\begin{aligned} E\left[\exp\left((\gamma-1) p_{F,T}\right)\right] =& E\left[I\left(r_{F,t} \geq \ln\kappa + \eta\right)\right] \\ &+ E\left[\exp\left((\gamma-1)\left(\ln\kappa + \eta - r_{F,T}\right)\right) I\left(r_{F,t} < \ln\kappa + \eta\right)\right] \end{aligned} \quad \text{(A.8)}$$

In order to compute the above expectations and for some later results in this Appendix, we introduce the following lemma.

Lemma A.1 *Let X and Y be two normally distributed random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2, and covariance σ_{XY}. Then, we have*

$$\begin{aligned} &E\left[\exp\left(uX + vY\right) I\left(X < x\right)\right] \\ &\quad = \exp\left(u\mu_X + v\mu_Y + \frac{1}{2}\left(u^2\sigma_X^2 + 2uv\sigma_{XY} + v^2\sigma_Y^2\right)\right) \Phi\left(\frac{x-\mu_X}{\sigma_X} - u\sigma_X - v\frac{\sigma_{XY}}{\sigma_X}\right), \end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Define the function

$$M\left(u, z; x\right) \equiv E\left[\exp\left(ur_{F,T} + z^\top r_T\right) I\left(r_{F,T} < x\right)\right] \ , \quad \text{(A.9)}$$

and note that $z^\top r_T \sim N\left(z^\top\mu, z^\top\Sigma z\right)$ and $Cov\left(r_{F,T}, z^\top r_T\right) = z^\top\Sigma w - z^\top\vartheta_L$. By applying

Lemma A.1,

$$M(u,z;x) = \exp\left(u\mu_F + z^\top\mu + \frac{1}{2}\left(u^2\sigma_F^2 + 2u\left(z^\top\Sigma w - z^\top\vartheta_L\right) + z^\top\Sigma z\right)\right) \times \Phi\left(\frac{x-\mu_F}{\sigma_F} - u\sigma_F - \frac{z^\top\Sigma w - z^\top\vartheta_L}{\sigma_F}\right) \ . \tag{A.10}$$

Using this result, (A.8) becomes

$$\begin{aligned} E\left[\exp\left((\gamma-1)\,p_{F,T}\right)\right] &= 1 - M(0,0;\ln\kappa+\eta) + \exp\left((\gamma-1)(\ln\kappa+\eta)\right)M(1-\gamma,0;\ln\kappa+\eta) \\ &= 1-\Phi(d_1) + S\Phi(d_2) \ , \end{aligned} \tag{A.11}$$

with

$$S \equiv \exp\left((\gamma-1)(\ln\kappa+\eta-\mu_F) + \frac{1}{2}(\gamma-1)^2\sigma_F^2\right) \ , \tag{A.12}$$

$$d_1 \equiv \frac{\ln\kappa+\eta-\mu_F}{\sigma_F} \quad \text{and} \quad d_2 \equiv d_1 + (\gamma-1)\,\sigma_F. \tag{A.13}$$

which leads to (12) in the main text.

A.2 Optimal Allocation Policy

Equation (A.6) defines an implicit function

$$G(w,\eta) = 0, \tag{A.14}$$

with

$$G(w,\eta) \equiv -\eta + \frac{1}{1-\gamma}\ln E\left[\exp\left((1-\gamma)\,r_{F,T}\right)\right] + \frac{1}{\gamma-1}\ln\left(\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left((\gamma-1)\,p_{F,T}\right)\right]-1\right)\right) . \tag{A.15}$$

If an optimal allocation policy w^* does exist, then it satisfies the necessary condition

$$\left.\frac{\partial \eta}{\partial w}\right|_{w=w^*} = 0. \tag{A.16}$$

Implicit differentiation of equation (A.14) implies that $\frac{\partial \eta}{\partial w} = -\frac{G_1(w,\eta)}{G_2(w,\eta)}$, where G_i is the partial derivative of G with respect to the i-th argument. Finally, $\frac{\partial \eta}{\partial w} = 0$ is equivalent to $G_1(w,\eta) = 0$. Computing $G_1(w,\eta)$ from (A.15), we have

$$G_1(w,\eta) = \frac{E\left[\exp\left((1-\gamma)\, r_{F,T}\right) \frac{\partial r_{F,T}}{\partial w}\right]}{E\left[\exp\left((1-\gamma)\, r_{F,T}\right)\right]} - \frac{\ell \kappa^{1-\gamma} E\left[\exp\left((\gamma-1)\, p_{F,T}\right) \frac{\partial p_{F,T}}{\partial w}\right]}{\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left((\gamma-1)\, p_{F,T}\right)\right] - 1\right)} \tag{A.17}$$

Note also that

$$\frac{\partial r_{F,T}}{\partial w} = \left(r_T - r_0 \iota + \frac{1}{2}\sigma^2\right) - \Sigma w \quad \text{and} \quad \frac{\partial p_{F,T}}{\partial w} = -\frac{\partial r_{F,T}}{\partial w} I\left(r_{F,T} < \ln\kappa + \eta\right) . \tag{A.18}$$

Using (A.18), in (A.17) yields

$$\begin{aligned} G_1(w,\eta) =& \frac{E\left[\exp\left((1-\gamma)\, r_{F,T}\right) r_T\right]}{E\left[\exp\left((1-\gamma)\, r_{F,T}\right)\right]} - r_0\iota + \frac{1}{2}\sigma^2 - \Sigma w \\ &+ \frac{\ell\kappa^{1-\gamma} E\left[\exp\left((\gamma-1)\left(\ln\kappa + \eta - r_{F,T}\right)\right)\left(r_T - r_0\iota + \frac{1}{2}\sigma^2 - \Sigma w\right) I\left(r_{F,T} < \ln\kappa + \eta\right)\right]}{\theta - \ell\kappa^{1-\gamma}\left(E\left[\exp\left((\gamma-1)\, p_{F,T}\right)\right] - 1\right)} \end{aligned} \tag{A.19}$$

The partial derivative of $M(u,z;x)$ with respect to its second argument is

$$\begin{aligned} M_2(u,z;x) &= E\left[\exp\left(u r_{F,T} + z^\top r_T\right) r_T I\left(r_{F,T} < x\right)\right] \\ &= \Bigg[\left(\mu + u\left(\Sigma w - \vartheta_L\right) + \Sigma z\right)\Phi\left(\frac{x-\mu_F}{\sigma_F} - u\sigma_F - \frac{z^\top\Sigma w - z^\top\vartheta_L}{\sigma_F}\right) \\ &\qquad -\frac{1}{\sigma_F}\left(\Sigma w - \vartheta_L\right)\phi\left(\frac{x-\mu_F}{\sigma_F} - u\sigma_F - \frac{z^\top\Sigma w - z^\top\vartheta_L}{\sigma_F}\right)\Bigg] \\ &\qquad \times \exp\left(u\mu_F + z^\top\mu + \frac{1}{2}\left(u^2\sigma_F^2 + 2u\left(z^\top\Sigma w - z^\top\vartheta_L\right) + z^\top\Sigma z\right)\right), \end{aligned} \tag{A.20}$$

where $\phi(\cdot)$ is the standard normal probability distribution function.

Using this result,

$$\frac{E\left[\exp\left((1-\gamma)\,r_{F,T}\right)r_T\right]}{E\left[\exp\left((1-\gamma)\,r_{F,T}\right)\right]} = \frac{M_2\left(1-\gamma,0;\infty\right)}{M\left(1-\gamma,0;\infty\right)} = \mu + (1-\gamma)\left(\Sigma w - \vartheta_L\right) \tag{A.21}$$

and

$$\begin{aligned} &E\left[\exp\left((\gamma-1)\left(\ln\kappa+\eta-r_{F,T}\right)\right)\left(r_T - r_0\iota + \frac{1}{2}\sigma^2 - \Sigma w\right) I\left(r_{F,T} < \ln\kappa+\eta\right)\right] \\ &\quad = \exp\left((\gamma-1)\left(\ln\kappa+\eta\right)\right) M_2\left(1-\gamma,0;\ln\kappa+\eta\right) \\ &\qquad + \exp\left((\gamma-1)\left(\ln\kappa+\eta\right)\right) M\left(1-\gamma,0;\ln\kappa+\eta\right)\left(-r_0\iota+\frac{1}{2}\sigma^2-\Sigma w\right) \\ &\quad = S\left[\mu + (1-\gamma)\left(\Sigma w-\vartheta_L\right)\Phi\left(d_2\right) - \frac{1}{\sigma_F}\left(\Sigma w - \vartheta_L\right)\phi\left(d_2\right)\right] \\ &\qquad + S\Phi\left(d_2\right)\left(-r_0\iota+\frac{1}{2}\sigma^2-\Sigma w\right) \;. \end{aligned} \tag{A.22}$$

These last two results can be substituted into (A.19). Then, after some algebraic manipulation, $G_1(w,\eta)=0$ leads to

$$w = \frac{1}{\tilde{\gamma}}\left(\Sigma^{-1}\left(\mu - r_0\iota + \frac{1}{2}\sigma^2\right)\right) + \left(1-\frac{1}{\tilde{\gamma}}\right)\left(\Sigma^{-1}\vartheta_L\right) \;, \tag{A.23}$$

with

$$\tilde{\gamma} = \gamma + \frac{1}{\sigma_F}\cdot\frac{\ell\kappa^{1-\gamma}S\phi\left(d_2\right)}{\theta+\ell\kappa^{1-\gamma}\Phi\left(d_1\right)} \;. \tag{A.24}$$

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Table 1: Return assumptions

Table 1 reports annualized expected returns, volatilities, and correlations of bonds and equities, which are total returns of the Ibbotson US Long-Term Corporate Debt Index and the S&P 500 Index between January 1952 and December 2011. All the values are taken from Exhibit 1 of Ang et al. (2013, p. 76).

			Correlations		
	Mean	Volatility	Bond	Equity	Liability
Bond	6.92%	8.60%	1.00		
Equity	11.04%	14.69%	0.25	1.00	
Liability	6.92%	10.00%	0.98	0.35	1.00
Risk-Free	4.00%				

Figure 1: Optimal choice of a disappointment averse manager

Figure 1 describes the optimal choice of the manager. The preference parameters used are $\gamma = 5$, $\kappa = 1$, and the different line types correspond to different values of ℓ. The panels show the log certainty equivalent η (in Panel A), the expectation $E\left[\exp\left(\left(\gamma-1\right)p_{F,T}\right)\right]-1$ (in Panel B), and the downside risk penalty $\frac{1}{\gamma-1}\ln\left(\theta-\ell\kappa^{1-\gamma}\left(E\left[\exp\left(\left(\gamma-1\right)p_{F,T}\right)\right]-1\right)\right)$ (in Panel C) corresponding to portfolios of the form $w = \frac{1}{\tilde{\gamma}}w^{\mathbf{MV}}+\left(1-\frac{1}{\tilde{\gamma}}\right)w^{\mathbf{LH}}$. The weight attached to the mean-variance portfolio, $\frac{1}{\tilde{\gamma}}$, is varied along the horizontal axis in each panel. The markers in Panel A correspond to the optimal portfolios, where the certainty equivalent of the manager is maximized. At these portfolios the value of $\tilde{\gamma}$ is given by (17).

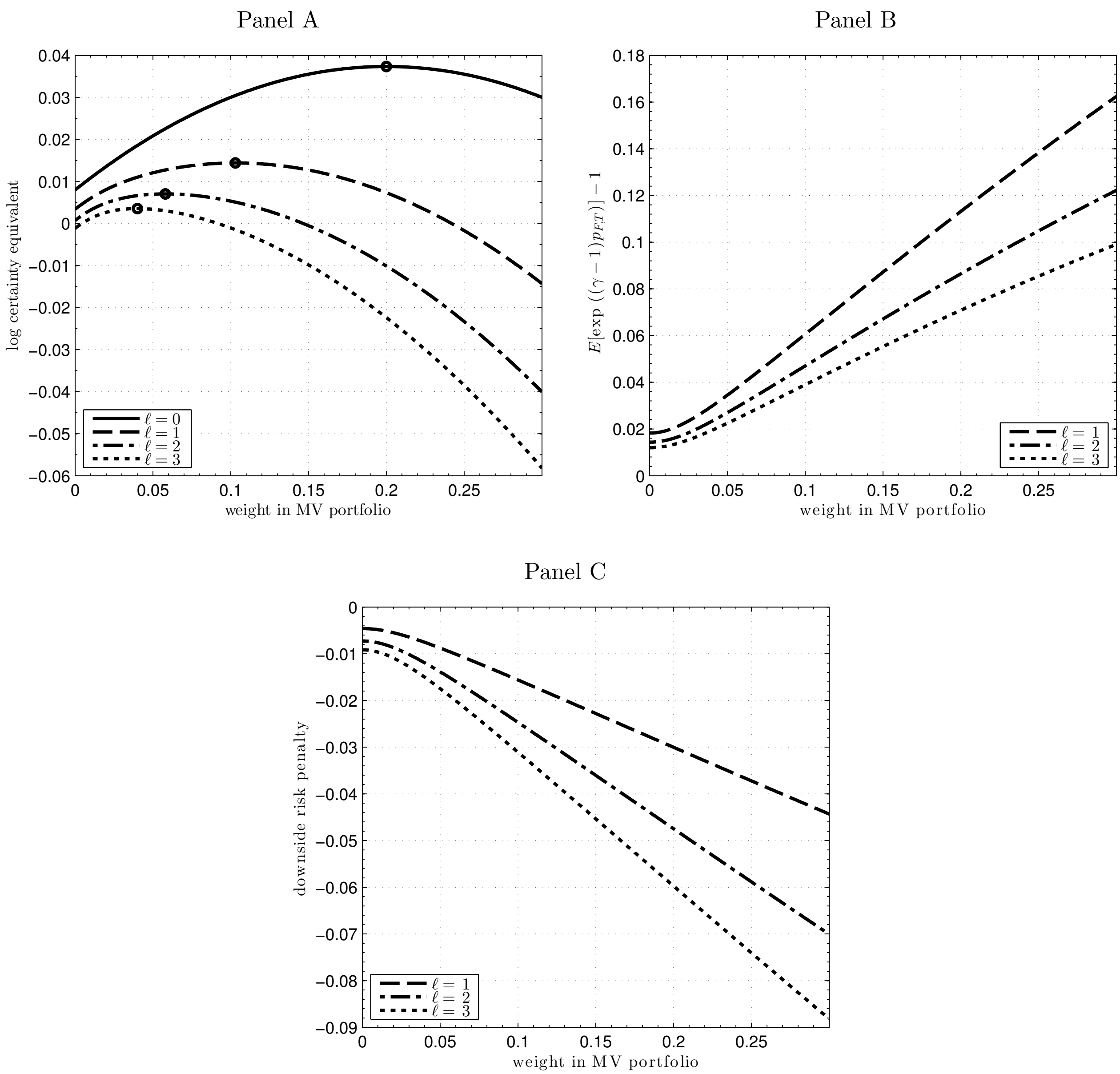

Figure 2: Optimal allocation policy for different degrees of disappointment aversion

Figure 2 shows optimal asset allocation strategies for different degrees of disappointment aversion. The preference parameters used are $\gamma = 5$, $\kappa = 1$, and ℓ varies along the horizontal axis in each panel. Panel A presents the inverse of the effective risk aversion $\frac{1}{\tilde{\gamma}}$, which is equal to the weight assigned to the mean-variance portfolio in the optimal asset allocation rule (15). Panel B presents the corresponding weights in the stock and the bond.

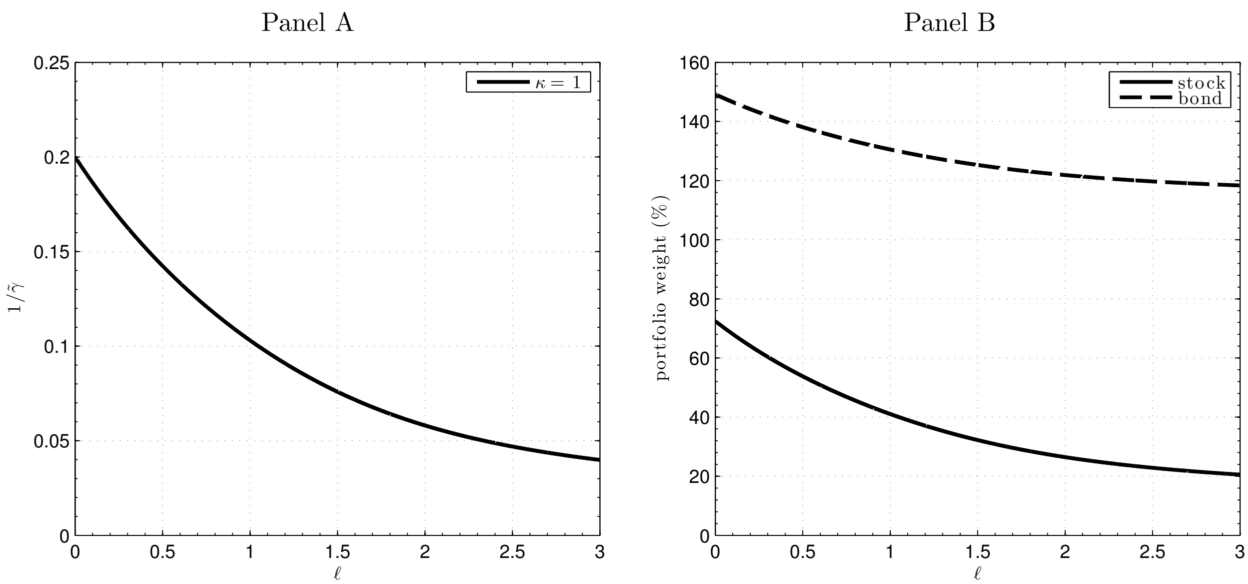

Figure 3: Optimal allocation policy for different disappointment thresholds

Figure 3 shows optimal asset allocation strategies for different disappointment thresholds. The preference parameters used are $\gamma = 5$, $\ell = 2$, and κ varies along the horizontal axis in each panel. Panel A presents the inverse of the effective risk aversion $\frac{1}{\tilde{\gamma}}$, which is equal to the weight assigned to the mean-variance portfolio in the optimal asset allocation rule (15). Panel B presents the corresponding disappointment threshold $\ln \kappa + \eta$.