Solving Models with Disappointment Aversion[*]

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August 30, 2014

Abstract

Asymmetric preferences have become an important ingredient to the empirical and theoretical asset pricing literature. Models with generalized disappointment aversion naturally incorporate downside risk into an axiomatic model of decision theory, but they cannot be solved anlytically without a state-space approximation of the economy. We propose a solution method to dynamic general equilibrium models with generalized disappointment averse preferences and continuous state endowment dynamics. We apply the framework to the term structure of interest rates and show that the model generates an upward sloping term structure of nominal interest rates, a downward sloping term structure of real interest rates, and that it accounts for the failure of the expectations hypothesis. The success comes from the ability of the model to endogenously generate strong countercyclical risk aversion. The key ingredients are disappointment averse preferences, preference for early resolution of uncertainty, and a parsimonious endowment economy with three state variables: time-varying macroeoconomic uncertainty, time-varying expected inflation and inflation uncertainty.

Keywords: Generalized Disappointment Aversion, Expectations Hypothesis, Numerical solution methods, Term Structure of Interest Rates

JEL Classification: C61, C63, E43, E44, G11, G12, G13

[*]Augustin acknowledges financial support from the Institute of Financial Mathematics of Montreal (IFM2).

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1 Introduction

We propose a solution method for preference-based models with asymmetric preferences and continuous-state endowment dynamics. Asymmetric preferences over losses and gains influence investors to make choices that are inconsistent with the predictions of expected utility theory. Such a behavior is rationally explained through a relatively greater risk aversion to disappointing events.[1] Models that incorporate such asymmetries into preferences have proved successful in rationalizing several asset pricing anomalies. Routledge and Zin (2010) generalize the disappointment averse preference framework and show how the endogenous variation in disappointment probability produces countercycical risk aversion, a necessary ingredient for resolving asset pricing puzzles. Bonomo et al. (2011) show that persistent fluctuations in macroeconomic uncertainty and asymmetric preferences are sufficient to generate realistic return predictability patterns, and to explain first and second moments of asset returns and price-dividend ratios. Augustin and Tédongap (2014) illustrate how asymmetric preferences help improve the quantitative implications of the conditional moments of credit default swap spreads, while Campanale et al. (2010) study the implications of disappointment averse preferences for asset prices in a production economy.[2]

The discontinuity in the preferences, characterized through a kink in the indifference curves, complicates the solutions of analytical asset pricing formulas. As a consequence, a common denominator to all prior references is a discrete state space approximation of the economy. While discrete regime-switching models approximate continuous processes well in population (Timmermann (2000)), they are not useful to study the properties of highly persistent processes in small samples. However, asset-pricing frameworks with recursive utility

[1] Gul (1991) introduces an axiomatic model of decision making under uncertainty with disappointment aversion that can rationalize the Allais paradox, i.e. realized choices that are inconsistent with expected utility theory.

[2] Other applications include Dolmas (2013), who combines disappointment aversion with rare disasters, Farago and Tédongap (2014), who use the framework to decompose asset risk premia into regular and downside risk premia, and Delikouras (2014), who applies preferences with disappointment to explain the credit spread puzzle.

do often rely on highly persistent processes, in particular for the dynamics of consumption growth volatility. The incorporation of asymmetric preferences into asset pricing models through the mechanism of generalized disappointment aversion is growing. We thus find it necessary to suggest a method that allows solving such models accurately and efficiently when the state of the economy is continuous, consistent with empirically observed dynamics.

We present a framework that allows for an accurate and efficient solution to asset prices when preferences feature non-linearities and the endowment dynamics are continuous in state-dependent outcomes. The model must be solved numerically, and the main challenge with numerical integration is computational complexity. Thus we propose a parsimonious model of real aggregate consumption growth with only one single state variable, the volatility of aggregate consumption growth. In addition, we assume that realized consumption growth and economic uncertainty are impacted by the same shock. This allows us to limit the resolution of asset prices to a one-dimensional numerical integration.

We apply the framework to the term structure of real and nominal interest rates and specify explicit dynamics for real growth. We also specify an exogenous process for inflation, which is necessary to price nominal assets. Consumption growth is non-predictable and features an affine GARCH model for macroeconomic uncertainty. Realized inflation has a time-varying mean, and the innovations in both realized and expected inflation are perfectly positively correlated, making realized inflation an ARMA(1,1) process. Like macroeconomic uncertainty, inflation uncertainty follows affine GARCH dynamics.[3] While inflation innovations are not allowed to affect future consumption growth, innovations in consumption growth can affect realized and expected inflation. Nominal prices thus rely on only three state variables: time-varying macroeoconomic uncertainty, time-varying expected inflation and inflation uncertainty.

The model is able match an upward sloping term structure of nominal interest rates and volatilities estimated in the data. An upward sloping nominal yield curve is obtained if infla-

[3]Realized inflation thus follows ARMA(1,1)-GARCH(1,1) dynamics.

tion is negatively correlated with innovations in aggregate consumption growth and the agent prefers early resolution of uncertainty. If consumption is negatively correlated with expected inflation, then agents will borrow from future consumption buy issuing bonds. This drives down nominal bond prices and increases nominal yields. Long-term bonds are, however, less sensitive to expected inflation shocks than short-term bonds. On the other hand, bond yields respond negatively to a rise in inflation uncertainty, and more so for longer-maturity bonds. This suggests a flight-to-quality effect in response to nominal uncertainty. Finally, the effect of real uncertainty on nominal bond prices depends on the asset horizon. For short-term bonds, a flight-to quality effect dominates whereby higher consumption volatility lowers nominal yields. In contrast, for long-term bonds, the intertemporal substitution effect dominates, as agents start to front-load their consumption, which depresses nominal bond prices and increases yields. The term structure of real interest rates is negative, consistent with the intuition that inflation-indexed bonds provide a hedge against future consumption. Thus, agents are willing to pay a premium to hold such assets, which implies a negative risk premium.

Moreover, the model quantitatively accounts for the failure of the expectations hypothesis. We replicate different versions of the regressions that have confirmed the existence of predictability in bond returns using a simulation of 300,000 months of data. We quantitatively match the tent-shaped pattern of the projection of holding period returns on the single Cochrane and Piazzesi (2005) factor, the Fama and Bliss (1987) regressions of holding period returns on forward-spot spreads, the Campbell and Shiller (1991) regressions of changes in long rate spreads on yield-spot spreads, and the Dai and Singleton (2002a) regressions of adjusted changes in long rate spreads on yield-spot spreads.

The success of the model relies partly on the model's ability to generate both time-varying prices and quantities of risk. This is a desirable feature for equilibrium models, as Le and Singleton (2013) have pointed out. The model endogenously generates time-varying prices of risk through disappointment averse preferences, based on the work of Gul (1991).

Outcomes below the certainty equivalent of future lifetime utility are disappointing. As the certainty equivalent evolves dynamically, the pricing kernel implied by the disappointment averse preference exhibits endogenously time-varying market prices of risk. The kink in the indifference curve, which arises from the asymmetry in preferences, also introduces a volatility of the pricing kernel that is at least as large as that of an investor with symmetrically recursive preferences of Epstein and Zin (1989), as we theoretically show. These features generate strongly time-varying and countercyclical risk aversion that is able to quantitatively match the predictability patterns in nominal bond returns.

2 Model

This section describes the generalized disappointment aversion (GDA) preferences of Routledge and Zin (2010) and characterizes the general framework for the endowment dynamics.

2.1 Preferences and Stochastic Discount Factor

The representative agent exhibits aversion for disappointing outcomes. Such preferences are based on the work of Gul (1991) and have been generalized by Routledge and Zin (2010).[4] In the spirit of Epstein and Zin (1989) and Weil (1989), the investor derives utility V_t recursively from a weighted average of current and future consumption

$$\begin{aligned} V_t &= \left\{ (1-\delta)\, C_t^{1-\frac{1}{\psi}} + \delta \left[\mathcal{R}_t\left(V_{t+1}\right)\right]^{1-\frac{1}{\psi}} \right\}^{\frac{1}{1-\frac{1}{\psi}}} \quad \text{if } \psi \neq 1 \\ &= C_t^{1-\delta} \left[\mathcal{R}_t\left(V_{t+1}\right)\right]^{\delta} \quad \text{if } \psi = 1, \end{aligned} \tag{2}$$

where C_t denotes the level of current consumption and $\mathcal{R}_t\left(V_{t+1}\right)$ is the certainty equivalent of next period lifetime utility, summarizing the utility over all future consumption streams. The time preference parameter is given by $0 < \delta < 1$, and $\psi > 0$ characterizes the elasticity

[4] Note that this preference framework is based on axiomatic decision theory.

of intertemporal substitution.

As the agent is averse towards disappointing outcomes, she has asymmetric preferences over good and bad outcomes. More specifically, with GDA preferences, the risk-adjustment function $\mathcal{R}(.)$ is implicity defined by

$$\frac{\mathcal{R}^{1-\gamma}-1}{1-\gamma}=\int_{-\infty}^{\infty}\frac{V^{1-\gamma}-1}{1-\gamma}dF\left(V\right)-\ell\int_{-\infty}^{\kappa\mathcal{R}}\left(\frac{\left(\kappa\mathcal{R}\right)^{1-\gamma}-1}{1-\gamma}-\frac{V^{1-\gamma}-1}{1-\gamma}\right)dH\left(V\right), \qquad (3)$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $H(\cdot)$ represents the cumulative distribution function of the random outcome.[5] The coefficient of generalized disappointment aversion $0 < \kappa \leq 1$ defines the fraction of the certainty equivalent below which an outcome is considered to be disappointing, while the coefficient of disappointment aversion $\ell \geq 0$ defines by how much utility is reduced in disappointing states. With ℓ equal to zero, the model nests the symmetric Kreps and Porteus (1978) certainty equivalent $\mathcal{R}$ and V_t is restored to be the standard Epstein and Zin (1989) recursive utility. When $\ell > 0$, outcomes below a fraction κ of the certainty equivalent $\mathcal{R}$ lower the certainty equivalent by an amount modulated by ℓ.

From (3) it follows that the risk-adjusted future lifetime utility may be rewritten as

$$\mathcal{R}_t\left(V_{t+1}\right)=\left(E_t\left[\frac{1+\ell I\left(V_{t+1}<\kappa\mathcal{R}_t\left(V_{t+1}\right)\right)}{1+\ell\kappa^{1-\gamma}E_t\left[I\left(V_{t+1}<\kappa\mathcal{R}_t\left(V_{t+1}\right)\right)\right]}V_{t+1}^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}, \qquad (4)$$

where $I(\cdot)$ is an indicator function defined to be 1 if the condition is met and 0 otherwise. Hansen et al. (2007) derive the stochastic discount factor $M_{t,t+1}$ in terms of the continuation value of utility of consumption

$$M_{t,t+1}=M_{t,t+1}^{*}\left(\frac{1+\ell I\left(V_{t+1}<\kappa\mathcal{R}_t\left(V_{t+1}\right)\right)}{1+\ell\kappa^{1-\gamma}E_t\left[I\left(V_{t+1}<\kappa\mathcal{R}_t\left(V_{t+1}\right)\right)\right]}\right) \qquad (5)$$

[5] Note that when $\gamma = 1$, we have that $\ln\mathcal{R}=\int_{-\infty}^{\infty}\ln V dH\left(V\right)-\ell\int_{-\infty}^{\kappa\mathcal{R}}\left(\ln\left(\kappa\mathcal{R}\right)-\ln V\right)dH\left(V\right)$.

where

$$M^*_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{V_{t+1}}{\mathcal{R}_t \left(V_{t+1} \right)} \right)^{\frac{1}{\psi} - \gamma}. \tag{6}$$

2.2 Economy

Solving models of generalized disappointment aversion with continuous endowment dynamics requires numerical solution methods. In practice, this implies that solutions to asset prices involve an integration over the support of each independent source of risk. This computational complexity makes a parsimonious model attractive and desirable. We thus assume that the state of the real economy is characterized by a single state variable σ_t, the volatility of aggregate consumption growth.[6] The existence of fluctuations in macroeconomic uncertainty is now a well-established fact (Kandel and Stambaugh (1990) and Stock and Watson (2002)) and its importance for asset prices has been demonstrated, among many others, by Bansal et al. (2005) and Lettau et al. (2006).

Let $\mathcal{O}(\cdot;\cdot)$ and $\mathcal{S}(\cdot;\cdot)$ be two generic functions describing, respectively, the observation and state equations in a state-space system, where $\mathcal{S}(\cdot;\cdot)$ is non-negative. Then we generally define the dynamics of real aggregate consumption growth rate as

$$\begin{aligned} \Delta c_{t+1} &= \mathcal{O}\left(u_{t+1}; \sigma_t^2\right) \\ \sigma_{t+1}^2 &= \mathcal{S}\left(u_{t+1}; \sigma_t^2\right) \end{aligned} \tag{7}$$

where u_{t+1} is an independent and identically distributed standard normal shock, which impacts both real growth and macroeconomic uncertainty.

The advantage of a single source of risk affecting both the output and state equation is that it restricts the resolution of asset prices to a one-dimensional integral calculation. While it is feasible to introduce an independent source of risk for consumption volatility,

[6] In fact, Bonomo et al. (2011) show that a model with generalized disappointment aversion without persistent fluctuations in the mean of aggregate consumption growth improves empirical return predictability patterns over a specification with recursive utility and long-run risk in expected consumption growth.

this would require a two-dimensional integration to solve for asset prices. Higher-order integration significantly reduces the numerical precision of results.[7] The diversity of possible integration regions and singularities for multivariate functions is daunting. As a general rule, it is not possible to obtain the same accuracy with higher-dimensional integrals as with one-dimensional integrals for reasonable computing times.

3 Solving The Model

Whatever difficulty we face in solving the model lies in finding an accurate solution to the welfare valuation ratios V_t/C_t and $\mathcal{R}_t(V_{t+1})/C_t$, and, to a lesser extent, the real risk-free rate. These are functions of macroeconomic uncertainty σ_t^2, the single state variable of the real economy. This section shows how to explicitly solve for the welfare valuation ratios, which allow us to derive the probability of disappointment and the stochastic discount factor.

Given a specification of the endowment process defined in equation (7), we can solve for the welfare valuation ratios V_t/C_t and $\mathcal{R}_t(V_{t+1})/C_t$

$$\frac{V_t}{C_t} = G^V\left(\sigma_t^2\right) \quad \text{and} \quad \frac{\mathcal{R}_t(V_{t+1})}{C_t} = G^{\mathcal{R}}\left(\sigma_t^2\right), \tag{8}$$

which are functions of the volatility of aggregate consumption growth σ_t^2. From the recursion (2), it follows that $G^V = F\left(G^{\mathcal{R}}\right)$ where

$$\begin{aligned} F(X) &= \left\{(1-\delta) + \delta X^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-\frac{1}{\psi}}} \quad \text{if } \psi \neq 1, \\ &= X^{\delta} \quad \text{if } \psi = 1 \end{aligned} \tag{9}$$

for any positive real number X. Hence the first step is to derive a solution to $G^{\mathcal{R}}$, the ratio of the certainty equivalent of future lifetime utility to the current consumption level. Define

[7] See Smyth (1998) and references therein.

the function $Z^{\mathcal{R}}(\cdot;\cdot)$

$$Z^{\mathcal{R}}(\cdot;\cdot) = F\left(G^{\mathcal{R}}\left(\mathcal{S}(\cdot;\cdot)\right)\right)\exp\left(\mathcal{O}(\cdot;\cdot)\right). \tag{10}$$

It follows from the expression of the certainty equivalent (4) that

$$\begin{aligned} G^{\mathcal{R}}\left(\sigma_t^2\right) &= \left(\frac{E_t\left[\left(1+\ell I\left(Z^{\mathcal{R}}\left(u_{t+1};\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right)\left(Z^{\mathcal{R}}\left(u_{t+1};\sigma_t^2\right)\right)^{1-\gamma}\right]}{1+\ell\kappa^{1-\gamma}E_t\left[I\left(Z^{\mathcal{R}}\left(u_{t+1};\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right]}\right)^{\frac{1}{1-\gamma}} \\ &= \left(\frac{\int\limits_{-\infty}^{\infty}\left(1+\ell I\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right)\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right)\right)^{1-\gamma}\phi(u)\,du}{1+\ell\kappa^{1-\gamma}\int\limits_{-\infty}^{\infty}I\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\phi(u)\,du}\right)^{\frac{1}{1-\gamma}} \end{aligned} \tag{11}$$

where $\phi(\cdot)$ is the probability density function of the standard normal distribution. We can solve equation (11) recursively using numerical integration.[8] This method is similar to the one used by Campbell and Cochrane (1999) to solve for the price-consumption ratio in the external habit model, and referred to in Wachter (2005) as the *fixed-point* method.[9]

We initiate the recursion by conjecturing a solution $G_0^{\mathcal{R}}\left(\sigma_t^2\right)$. $G_1^{\mathcal{R}}\left(\sigma_t^2\right)$ is then obtained on a grid of values for σ_t^2, as

$$G_1^{\mathcal{R}}\left(\sigma_t^2\right) = \left(\frac{\int\limits_{-\infty}^{\infty}\left(1+\ell I\left(Z_0^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G_0^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right)\left(Z_0^{\mathcal{R}}\left(u;\sigma_t^2\right)\right)^{1-\gamma}\phi(u)\,du}{1+\ell\kappa^{1-\gamma}\int\limits_{-\infty}^{\infty}I\left(Z_0^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G_0^{\mathcal{R}}\left(\sigma_t^2\right)\right)\phi(u)\,du}\right)^{\frac{1}{1-\gamma}} \tag{12}$$

where $Z_0^{\mathcal{R}}\left(u;\sigma_t^2\right) = F\left(G_0^{\mathcal{R}}\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\right)\exp\left(\mathcal{O}\left(u;\sigma_t^2\right)\right)$.

More generally, for any k, given a vlue for $G_k^{\mathcal{R}}\left(\sigma_t^2\right)$, we obtain the value of $G_{k+1}^{\mathcal{R}}\left(\sigma_t^2\right)$ on

[8]In order to solve the integrals, we need to choose a numerical integration routine and bounds on the normal shock. We choose the adadptive Simpson quadrature method and the integral is bounded by -8 and +8 standard deviations.

[9]Wachter (2005) compares the advantages of the *fixed point* and *series* methods for the speed of convergence of the solution to the term structure of interest rates. Such a comparison is not possible as the welfare valuation ratios follow a non-linear recursion which cannot be expressed as a sum of recursive terms.

a grid of values for σ_t^2, as

$$G_{k+1}^{\mathcal{R}}\left(\sigma_t^2\right)=\left(\frac{\int\limits_{-\infty}^{\infty}\left(1+\ell I\left(Z_k^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G_k^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right)\left(Z_k^{\mathcal{R}}\left(u;\sigma_t^2\right)\right)^{1-\gamma}\phi\left(u\right)du}{1+\ell\kappa^{1-\gamma}\int\limits_{-\infty}^{\infty}I\left(Z_k^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G_k^{\mathcal{R}}\left(\sigma_t^2\right)\right)\phi\left(u\right)du}\right)^{\frac{1}{1-\gamma}} \tag{13}$$

where $Z_k^{\mathcal{R}}\left(u;\sigma_t^2\right)=F\left(G_k^{\mathcal{R}}\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\right)\exp\left(\mathcal{O}\left(u;\sigma_t^2\right)\right)$. The iterations are repeated until

$$\left\|1-\frac{G_{k+1}^{\mathcal{R}}}{G_k^{\mathcal{R}}}\right\|<10^{-4}.$$

The recursion yields a fixed point, which is the solution $G^{\mathcal{R}}\left(\sigma_t^2\right)$ to equation (11). At each step of the recursion, we need to evaluate the function obtained in the previous iteration at a set of points $\mathcal{S}\left(u_{t+1};\sigma_t^2\right)$ (where $\{u_{t+1}\}$ is determined by the numerical integration routine) for each value of σ_t^2. Typically, these points lie outside the predefined grid of values for σ_t^2. Thus, we apply a log-linear interpolation method to evaluate $G_k^{\mathcal{R}}$ at these points, assuming that $\ln G_k^{\mathcal{R}}\left(\sigma_t^2\right)$ is linear in σ_t^2.[10] Given the solutions $G^{\mathcal{R}}$, we can derive the solution of $G^V=F\left(G^{\mathcal{R}}\right)$. It is then straightforward to compute the disappointment probability ξ_t as

$$\begin{aligned}\xi_t\equiv\xi\left(\sigma_t^2\right)&=E_t\left[I\left(Z^{\mathcal{R}}\left(u_{t+1};\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right]\\&=\int\limits_{-\infty}^{\infty}I\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\phi\left(u\right)du.\end{aligned} \tag{14}$$

It follows from equation (8) that the real stochastic discount factor is given by

$$\begin{aligned}M_{t,t+1}=\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}&\left(\frac{G^V\left(\sigma_{t+1}^2\right)}{G^{\mathcal{R}}\left(\sigma_t^2\right)}\right)^{\frac{1}{\psi}-\gamma}\\&\times\left(\frac{1+\ell I\left(G^V\left(\sigma_{t+1}^2\right)\left(C_{t+1}/C_t\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)}{1+\ell\kappa^{1-\gamma}E_t\left[I\left(G^V\left(\sigma_{t+1}^2\right)\left(C_{t+1}/C_t\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right]}\right),\end{aligned} \tag{15}$$

[10] This log-linear interpolation method is similar to the one used in Campbell and Cochrane (1999) to numerically solve the habit formation model.

which can be derived given the solutions to welfare valuation ratios V_t/C_t and $\mathcal{R}_t\left(V_{t+1}\right)/C_t$ based on equation (11), and the disappointment probability in equation (14) .

4 Application to the Term Structure of Interest Rates

We present an application of the general framework to the term structure of interest rates. We solve for the real and nominal yield curve and explain how the model matches an upward sloping term structure of nominal interest rates and volatilities consistent with the data.

4.1 Real Yield Curve

The price of a zero-coupon bond paying one unit of consumption n-periods ahead from now must statisfy the Euler equation

$$P_{n,t} = E_t\left[M_{t,t+n}\right]. \tag{16}$$

More generally, we can solve for the term structure of real interest rates recursively, given that the price $P_{n,t} \equiv P_n\left(\sigma_t^2\right)$ of the zero-coupon real bond that matures in n periods satisfies the recursion

$$P_{n,t} = E_t\left[M_{t,t+1}P_{n-1,t+1}\right], \tag{17}$$

with the initial condition $P_{0,t} = 1$. This price can be computed recursively using numerical integration as follows:

$$
\begin{aligned}
P_n\left(\sigma_t^2\right) = \delta \left(\frac{1}{G^{\mathcal{R}}\left(\sigma_t^2\right)}\right)^{\frac{1}{\psi}-\gamma} & \left(\frac{1}{1+\ell\kappa^{1-\gamma}\xi\left(\sigma_t^2\right)}\right) \\
\times \int_{-\infty}^{\infty} & \Big\{\exp\left(-\gamma\mathcal{O}\left(u;\sigma_t^2\right)\right)\left(G^V\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\right)^{\frac{1}{\psi}-\gamma} \\
& \times \left(1+\ell I\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right) < \kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right) P_{n-1}\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\Big\}\,\phi\left(u\right)du.
\end{aligned} \tag{18}
$$

Real yields to maturity n are defined as

$$
y_{n,t} = -\frac{1}{n}p_{n,t}, \tag{19}
$$

where $p_{n,t}$ is the logarithm of the real bond price.

4.2 Nominal Yield Curve

For the analysis of nominal prices, we need to specify a process for inflation π_{t+1}. More precisely, we assume that growth rates in prices have a time-varying mean and volatility. The dynamics of inflation π_{t+1} depend on two state variables, expected inflation, z_{t+1}, and price growth uncertainty, v_{t+1}, which impacts both expected and realized inflation, as described in the following system of equations

$$
\begin{aligned}
\pi_{t+1} &= \mu_\pi + z_t + \left(\nu_\pi\sigma_t u_{t+1} + \sqrt{v_t}\varepsilon_{t+1}\right) \\
z_{t+1} &= \phi_z z_t + \nu_z\left(\nu_\pi\sigma_t u_{t+1} + \sqrt{v_t}\varepsilon_{t+1}\right), \\
v_{t+1} &= \left(1-\phi_v\right)\mu_v - \nu_v + \left(\phi_v - \nu_v\beta_v^2\right)v_t + \nu_v\left(\varepsilon_{\pi,t+1} - \beta_v\sqrt{v_t}\right)^2,
\end{aligned} \tag{20}
$$

where ε_{t+1} is an independent and identically distributed standard normal shock, orthogonal to the shocks in consumption growth u_{t+1}.[11] Such a specification implies that innovations in expected and realized inflation are perfectly positively correlated. Shocks to aggregate consumption growth are thus allowed to impact expected and realized inflation. The parameter μ_π denotes the average inflation rate, $0 < \phi_z < 1$ modulates the persistence of expected price growth, ν_z determines the level of the expected inflation shock volatility, and $\nu_\pi < 0$ determines how uncertainty about real growth affects both realized and expected inflation. A negative value for ν_π imposes a negative correlation between innovations in consumption growth and realized inflation. The volatility process v_t is assumed to be the residual volatility of inflation volatility orthogonal to consumption volatility. The persistence of inflation uncertainty is parameterized through ϕ_v, μ_v defines the average level of inflation volatility and ν_v governs the residual volatility. The parameter β_v is a leverage coefficient, governing the correlation between inflation volatility innovations and both expected and realized price growth. The leverage parameter introduces skewness in low-frequency price growth, even though inflation has zero skewness at the single-period (monthly) horizon. Thus quarterly and yearly inflation rates are skewed.

Nominal assets can be priced by discounting nominal payoffs with the nominal stochastic discount factor, which is simply the difference between the real pricing kernel and the inflation rate. The logarithm of the nominal stochastic discount factor is thus expressed as

$$\ln M^{\$}_{t,t+1} = m^{\$}_{t,t+1} = \ln M_{t,t+1} - \pi_{t+1}. \tag{21}$$

Similar to the price of the real zero-coupon bond, the price of a nominal zero-coupon bond $P^{\$}_{n,t}$ paying one nominal unit of consumption n-periods ahead from now satisfies the

[11] We specify an exogenuous price growth process similar to the model for inflation in Wachter (2005) or Piazzesi and Schneider (2006). The affine dynamics for inflation uncertainty ensure that the numerical solution is restricted to a one-dimensional integration.

recursion

$$P_{n,t}^{\$} = E_t \left[M_{t,t+1}^{\$} P_{n-1,t+1}^{\$} \right], \tag{22}$$

with initial condition given by $P_{0,t}^{\$} = 1$. Equation (22) shows that, in contrast to real bonds, nominal bond prices are functions of more than one state variable. In addition to macroeconomic uncertainty, nominal bond prices also depend on expected inflation and inflation uncertainty. This could potentially complicate the solution to the model. However, we can use the law of iterated expectations and condition on the realizations of innovations to consumption growth to keep the numerical solution restricted to a one-dimensional integration. Given the assumption for the inflation dynamics, we can show that

$$P_{n,t}^{\$} = P_n^{\$}\left(\sigma_t^2\right) \exp\left(A_n^{\$} + B_{z,n}^{\$} z_t + B_{v,n}^{\$} v_t\right), \tag{23}$$

where the coefficients $A_n^{\$}$, $B_{z,n}^{\$}$ and $B_{v,n}^{\$}$ satisfy the recursions

$$\begin{aligned} A_n^{\$} &= -\mu_\pi + A_{n-1}^{\$} + \left(\left(1-\phi_v\right)\mu_v - \nu_v\right) B_{z,n-1}^{\$} - \frac{1}{2}\ln\left(1 - 2\nu_v B_{v,n-1}^{\$}\right) \\ B_{z,n}^{\$} &= \phi_z B_{z,n-1}^{\$} - 1 \\ B_{v,n}^{\$} &= \phi_v B_{v,n-1}^{\$} + \frac{\left(\nu_z B_{z,n-1}^{\$} - 1\right)^2}{2\left(1 - 2\nu_v B_{v,n-1}^{\$}\right)} \end{aligned} \tag{24}$$

with the initial conditions $A_0^{\$} = 0$, $B_{z,0}^{\$} = 0$ and $B_{v,0}^{\$} = 0$. The sequence $\left\{P_n^{\$}\left(\sigma_t^2\right)\right\}$ satisfies the recursion

$$P_n^{\$}\left(\sigma_t^2\right) = E_t\left[M_{t,t+1} P_{n-1}^{\$}\left(\sigma_{t+1}^2\right) \exp\left(\left(\nu_z B_{z,n-1}^{\$} - 1\right)\nu_\pi \sigma_t u_{t+1}\right)\right], \tag{25}$$

with the initial condition $P_0^{\$}\left(\sigma_t^2\right) = 1$. The recursion (25) has no closed-form solution and is solved by one-dimensional numerical integration over a grid of values for σ_t^2. Given $P_{n-1}^{\$}\left(\sigma_t^2\right)$, it follows that

$$P_n^{\$}\left(\sigma_t^2\right) = \delta\left(\frac{1}{G^{\mathcal{R}}\left(\sigma_t^2\right)}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{1}{1+\ell\kappa^{1-\gamma}\xi\left(\sigma_t^2\right)}\right)$$
$$\times\int_{-\infty}^{\infty}\left\{\exp\left(-\gamma\mathcal{O}\left(u;\sigma_t^2\right)+\left(\nu_z B_{z,n-1}^{\$}-1\right)\nu_\pi\sigma_t u\right)\left(G^V\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\right)^{\frac{1}{\psi}-\gamma}\right. \tag{26}$$
$$\left.\times\left(1+\ell I\left(Z^{\mathcal{R}}\left(u;\sigma_t^2\right)<\kappa G^{\mathcal{R}}\left(\sigma_t^2\right)\right)\right)P_{n-1}^{\$}\left(\mathcal{S}\left(u;\sigma_t^2\right)\right)\right\}\phi\left(u\right)du.$$

4.3 Explicit Model for Consumption Growth

For an empirical application of our theoretical framework, we need to specify an explicit process for the endowment economy, generally defined in equation (7). In other words, we need to provide a functional form for the state and observation equations $\mathcal{S}\left(u_{t+1};\sigma_t^2\right)$ and $\mathcal{O}\left(u_{t+1};\sigma_t^2\right)$. We model real aggregate growth to have a constant mean μ_c and affine GARCH dynamics for the variance process, following Heston and Nandi (2000).[12] Assuming a GARCH instead of, for example, stochastic volatility dynamics, avoids multi-dimensional integration when we compute asset prices numerically. In addition, it guarantees that volatility is a positive process. Formally, the real economy is defined as

$$\begin{aligned}\Delta c_{t+1} &= \mu_c + \sigma_t u_{t+1} \\ \sigma_{t+1}^2 &= \left(1-\phi_\sigma\right)\mu_\sigma - \nu_\sigma + \left(\phi_\sigma - \nu_\sigma\beta_\sigma^2\right)\sigma_t^2 + \nu_\sigma\left(u_{t+1}-\beta_\sigma\sigma_t\right)^2,\end{aligned} \tag{27}$$

where μ_σ denotes the unconditional mean, $0<\phi_\sigma<1$ modulates the persistence of macroeconomic uncertainty, β_σ determines the correlation between consumption growth and innovations in consumption volatility and ν_σ defines the sensitivity of growth rates to consumption

[12]Other consumption growth volatility dynamics, such as an exponential GARCH of Nelson (1991), would be consistent with our general framework. The solution method for other dynamics will remain similar as long as both realized and expected consumption growth are driven by the same source of shocks. These dynamics have recently been applied in Tédongap (2014) to study the implications for the cross-section of equity returns in a consumption-based equilibrium framework.

shocks. The unconditional variance is given by

$$\sigma_\sigma^2 = \frac{2\nu_\sigma^2 \left(1 + 2\beta_\sigma^2 \mu_\sigma\right)}{1 - \phi_\sigma^2}, \tag{28}$$

and the volatility process is well behaved if we impose the restrictions $(1 - \phi_\sigma)\,\mu_\sigma - \nu_\sigma \geq 0$ and $\phi_\sigma - \nu_\sigma \beta_\sigma^2 \geq 0$. Given μ_σ, $0 < \phi_\sigma < 1$ and σ_σ, the two non-negativity constraints imply that the volatility of volatility ν_σ and the leverage coefficient β_σ are given by

$$\nu_\sigma = \sqrt{\frac{\left(1 - \phi_\sigma^2\right) \sigma_\sigma^2}{2\left(1 + 2\beta_\sigma^2 \mu_\sigma\right)}} \quad \text{and} \quad \beta_\sigma^{\min} \leq |\beta_\sigma| \leq \beta_\sigma^{\max}$$

where

$$\begin{aligned} \beta_\sigma^{\min} &= \sqrt{\max\left(0, \frac{1}{2\mu_\sigma}\left(\frac{1}{2}\frac{1+\phi_\sigma}{1-\phi_\sigma}\frac{\sigma_\sigma^2}{\mu_\sigma^2} - 1\right)\right)} \\ \beta_\sigma^{\max} &= \sqrt{\frac{2\phi_\sigma^2 \mu_\sigma + \sqrt{2\phi_\sigma^2\left(2\phi_\sigma^2 \mu_\sigma^2 + \left(1 - \phi_\sigma^2\right)\sigma_\sigma^2\right)}}{\left(1 - \phi_\sigma^2\right)\sigma_\sigma^2}}. \end{aligned}$$

If $\beta_\sigma > 0$, then the two innovations are negatively correlated; they are positively correlated if $\beta_\sigma < 0$ and they are uncorrelated if $\beta_\sigma = 0$. The calibration of the model therefore requires a choice on the value of β_σ.[13,14]

4.4 Calibration and Data

The calibration of the model is summarized in Table 1 and is consistent with standard calibrations in long-run risk models, such as Bansal et al. (2012) or Bonomo et al. (2011). We calibrate the affine GARCH dynamics in equation (27) at the monthly decision interval to match the first and second moment of real annual US consumption growth from 1929 to 2011.

[13]Note that similar to the residual inflation volatility dynamics, the leverage coefficient introduces skewness in multi-period consumption growth rates, even though there is no skewness at the monthly horizon.

[14]The residual inflation volatility dynamics follow an analoguous GARCH recursion with similar parameter restrictions.

The mean of consumption growth is calibrated to $\mu_c = 0.0015$. The unconditional volatility of consumption growth, which is equal to $\sqrt{\mu_\sigma}$, is defined to be $\sqrt{\mu_\sigma} = 0.7305 \times 10^{-2}$. We set the persistence and the volatility of consumption volatility to $\phi_\sigma = 0.995$ and $\sigma_\sigma = 0.6263 \times 10^{-4}$. Given μ_σ, ϕ_σ and σ_σ, we choose $\beta_\sigma = \beta_\sigma^{\min}$ and ν_σ is defined in terms of the other parameter values.

The mean level of inflation μ_π is equal to 0.0030 and the inflation leverage on news ν_π is -0.1294, implying that realized inflation is negatively correlated with innovations in consumption growth. Expected inflation is highly persistent with a value of $\phi_z = 0.9840$ and the level of expected inflation volatility shock is $\nu_z = 0.3457$. Mean uncertainty in inflation is equal to $\mu_v = 6.3698 \times 10^{-7}$ and the persistence of inflation volatility is $\phi_v = 0.85$. Finally, the level of residual inflation volatility is $\nu_v = 9.5546 \times 10^{-8}$ and the leverage coefficient is given by $\beta_v = -2.9827 \times 10^{+3}$. Thus, inflation volatility is positively correlated with expected and realized price growth, which also implies a positive low-frequency inflation skewness.

Regarding preferences, we calibrate the intertemporal elasticity of substitution ψ at 1.5 and the constant relative risk aversion parameter γ at 2.5. This parameter configuration implies a preferences for early resolution of uncertainty, as is suggested empirically by the estimations in Bansal and Shaliastovich (2012) and Augustin and Tédongap (2014), among others. The disappointment aversion parameter ℓ is fixed at 1 and we set the threshold of the certainty equivalent below which outcomes become disappointing, κ, equal to 1.[15] Thus, in our benchmark calibration, any outcome below the certainty equivalent itself will reshuffle the state price probabilities. The subjective discount factor δ is equal to 0.9989.

The number of grid points may have a significant impact on the precision of the numerical solutions. We define a benchmark grid of 501 points for σ_t^2, and we will later evaluate the numerical precision of the solutions for alternative grid scenarios. The grid is defined in terms of the natural logarithm of σ_t^2, which has approximately a mean of $\mu_h = \ln \mu_\sigma - \sigma_h^2/2$

[15]Thus, for simplicity, we adopt a model of pure disappointment aversion without relying on the generalized version modeled by Routledge and Zin (2010).

and a standard deviation of $\sigma_h = \ln(1 + \sigma_\sigma^2/\mu_\sigma^2)$.[16] We use 334 logarithmically spaced points between $\mu_h - 7\sigma_h$ and μ_h, and 167 logarithmically spaced points between μ_h and $\mu_h + 5\sigma_h$. The lower segment is finer and includes values for volatility that are much closer to zero. This allows to better capture the nonlinear behavior of the welfare valuation ratio as volatility approaches zero. Given the assumed dynamics, more than 99.9% of the population distribution of $\ln \sigma_t^2$ lies between seven standard deviations below and five standard deviations above the mean.

To evaluate the implications of the model, we simulate a time-series of 300,000 months of data and compare the population moments to the sample data.[17] We use real data sampled at an annual frequency over the period 1929 to 2011. Data for consumption and price growth are taken from the Bureau of Economic Analysis National Income and Product Accounts Tables. To compare the model's solutions to the term structure of nominal interest rates, we use monthly Fama-Bliss discount bond prices from the CRSP US Treasury Database from January 1964 through December 2011. Fama-Bliss Discount Bonds Files contain artificial discount bonds with 1 to 5 years to maturity, constructed after first extracting the term structure from a filtered subset of the available bonds. This database is a refinement of the one used in Fama and Bliss (1987).

4.5 Model Solutions and Analysis

We start by discussing the model implications for real and nominal growth and then discuss the numerical solution to the welfare valuation ratios. We then report the results for the term structure of nominal interest rates and discuss the mechanism of the model.

[16]More precisely, μ_h and σ_h are approximate solutions to the mean and standard deviation of the logarithm of σ_t^2 when we apply a lognormal approximation to the volatility dynamics and if we ignore skewness. We emphasize that we obtain a grid for σ_t^2 by taking the exponent of the grid on $\ln \sigma_t^2$ element-by-element.

[17]A simulation of 300,000 observations is equivalent to the simulation of 100,000 quarters in Wachter (2005). In all simulations, we use a burn-in period of equal size, i.e. 300,000 months of data.

4.5.1 Real and Nominal Growth

We compare the time-averaged annualized moments from a simulated time-series of 300,000 monthly observations to the sample moments in the data, estimated using annual data over the period 1929 to 2011, in Table 2. Focusing first on the dynamics of consumption growth, the left panel shows a close fit between the (statistically significant) estimated moments and the model-implied unconditional moments. The mean growth rate is 1.97% in the data, while it is 1.79% in the model. Similarly, the comparison of the volatilities indicates 2.02% in the data versus a model-implied value of 2.08%. As expected with monthly non-predictable consumption growth dynamics, we obtain an annualized first-order autocorrelation of 0.24, which is a bit lower than the estimated value of 0.48.[18] Finally, we also obtain a reasonable fit for the skewness and kurtosis of the aggregate consumption growth dynamics.[19] Overall, the dynamics we have chosen for our empirical application reflect closely the distribution of aggregate consumption growth.

The right-hand panel in Table 2 compares the model-implied population values of the inflation process to the data estimates. Expected inflation is 3.17% (3.57%) in the sample (model), the annualized volatility is 3.29% (2.89%), the first-order autocorrelation cofficient is 0.83 (0.86), and the kurtosis is 8.64 (6.15). The skewness of inflation is estimated negatively at -0.80, although the estimate is not statistically different from zero. The model-implied value for inflation skewness is 1.34. This result arises because of the negative leverage parameter β_v in the inflation volatility dynamics, and it is consistent with those authors who argue for positive skewness in inflation.[20]

[18]It is possible to model the conditional mean of aggregate consumption growth as a function of consumption volatility. This would be one way to increase the autorrelation coefficient implied by the model.

[19]Note that the time aggregation introduces skewness at the annual horizon because of the leverage effect, even though the non-predictable consumption growth dynamics have zero skewness by construction at the monthly horizon.

[20]See for example Aizenman and Hausmann (1994) and Chaudhuri et al. (2013) and references therein.

4.5.2 Welfare Valuation Ratios

We next discuss the numerical solution to the utility-consumption ratio. A solution to this welfare valuation ratio is the primary input to obtain solutions to the stochastic discount factor, and therefore asset prices. Figure 1a plots the welfare valuation ratio $V_t/C_t = G^V(\sigma_t^2)$ as a function of consumption volatility σ_t^2 for our benchmark scenario with 501 grid points. The negative slope suggests that the ratio of utility to the level of consumption is decreasing for higher levels of consumption volatility. This is consistent with the notion that agents dislike macroeconomic uncertainty. To shed some light on the robustness of the numerical solution, we evaluate the solution to the welfare valuation ratio for different grids. We specify different densities ranging from ranging from the most coarse grid with 24 points to the finest grid with 501 points. We plot in Figure 1b the welfare valuation ratio $V_t/C_t = G^V(\sigma_t^2)$ as a function of consumption volatility for each of these grids. As becomes apparent in the figure, there is a significant difference in the results between the solution derived from the most coarse grid and the one with 76 points. The difference in solutions between the grid with 76 and 123 points is substantially smaller. On the other hand, there is hardly any improvement for the solution using 498 grid points over the solution using a grid of 249 points. This shows that the solution is accurate as soon as we use more than 250 grid points.

Another statistic of interest in the models with (generalized) disappointment aversion is the disappointment probability. Figure 1c reports the probability of disappointment $\xi(\sigma_t^2)$ as a function of consumption volatility σ_t^2. Without any macroeconomic uncertainty, the likelihood of disappointment is close to 26%. As consumption volatility increases, the disappointment probability shoots up to approximately 38% and stabilizes at relatively low levels of consumption volatility, around 1×10^{-5}.

4.5.3 The Term Structure of Real and Nominal Interest Rates

Given the solutions to the welfare valuation ratios and the disappointment probability, we are able to derive the solutions to real bond prices from equation (18). We plot in figure 1d the real yields $y_t^{(n)}$ as a function of consumption volatility for maturities one ($n = 1$) to five ($n = 5$) years. The model implies a downward sloping term structure of real interest rates. This is consistent with the intuition that inflation-indexed bonds represent a valuable hedge for long-term investors, which are willing to pay a premium to hold such assets (Campbell et al. (2009)). Payoffs of real bonds are fixed in consumption units, which are more highly valued when macreconomic uncertainty is high. As real bond returns are negatively correlated with the level of consumption and stock prices, they command a negative risk premium that increases with the asset horizon. This features are particularly true if shocks to macroeconomic uncertainty are persistent. In that case, positive innovations in economic uncertainty can lead to extended periods of slow growth, which increases real bond prices in recesions. A negative slope of the real yield curve is also consistent with a negative term structure of real interest rates found in the long-term UK data (Evans (1998)). For very high levels of macroeconomic uncertainty, the slope of the term structure of real interest rates becomes slightly more negative, as can be seen through the wider dispersion in the lines as we move closer to the right of the figure. At very high levels of consumption volatility (and for high asset horizons), real yields become negative, as expected. This is a common feature that arises in recursive utility models with long-run risks (see Bansal and Shaliastovich (2012)) when the parameter calibration implies a preference for early resolution of uncertainty.

We next turn to the main results of the application, the term structure of nominal bond prices. Table 2 reports the model-implied results for the term structure of nominal yields and the corresponding volatilities from the simulation of 300,000 months of data, corresponding to 25,000 years. The one-year and five-year nominal yield is 4.72% and 6.85%, compared to the values of 5.20% and 5.83% in the data. Thus, we generate a slightly steeper slope than in the data.[21] The volatility of nominal bond yields has also a similar magnitude to that found in the data.

4.5.4 A Discussion of the Model Mechanism

Equation (23) highlights that nominal bond prices are highly non-linear functions of three sources of risk: macroeconomic uncertainty, expected inflation and inflation uncertainty. In order to sharpen our intuition about the model's mechanism, we first plot in figures 1e and 1f the sensitivities of nominal bond yields $y_t^{(n)}$ to, respectively, expected inflation $(-B^{\$}_{z,n}/n)$ and inflation volatility $(-B^{\$}_{v,n}/n)$, as these maturity-dependent coefficients are known in closed form.[22] The loading of nominal bond yields to expected inflation is positive, implying that high expected inflation raises risk premia and increases nominal yield spreads. Nominal bond payoffs are fixed in terms of price levels, and therefore they pay off when expectations about future price growth are high. Innovations in consumption growth are negatively correlated with both realized and expected inflation. In other words, high expected inflation reflects a negative innovation to consumption growth. Thus, investors will issue bonds to borrow from future consumption. This depresses nominal bond prices and raises nominal yields. Note that $-B^{\$}_{z,n}/n$ is, a negative function of the asset horizon. In other words, short-run yields are relatively more sensitive to expected inflation shocks than longer-term yields.

The loading of nominal bond yields to inflation uncertainty is negative for all maturities. Thus, inflation volatility lowers nominal bond risk premia, which reflects a fligh-to-quality effect across all asset horizons. In times of high nominal uncertainty, investors develop a precautionary savings motive. This leads to them to buy nominal bonds, which raises their prices and lowers nominal yields. Given that $-B^{\$}_{v,n}/n$ is negative and has a negative slope too, the reduction in risk premia is relatively greater at longer horzions than at short-term maturities. Expected inflation and inflation uncertainty thus have opposing effects on

[21] We emphasize that the comparison is made between the results in population from a long simulation of 300,000 months of data to a small sample of approximately 47 years.

[22] Given that $y^{\$}_{n,t} = -\frac{1}{n}p^{\$}_{n,t}$, we have that $y^{\$}_{n,t} = -\frac{1}{n}p^{\$}_{n}\left(\sigma_t^2\right) - \frac{1}{n}A^{\$}_n - \frac{1}{n}B^{\$}_{z,n}z_t - \frac{1}{n}B^{\$}_{v,n}v_t$.

nominal bond prices and therm structure of nominal interest rates.

The third source of risk that impacts nominal bond yield prices is real uncertainty. In order to investigate the sensitivity of nominal bond yields to consumption volatility, we fix the values of expected inflation and inflation uncertainty at their long-run average values.[23] Conditional on these values, we plot in Figure 2 nominal bond yields for maturities one to five years as a function of consumption volatility. For one- and two-year bonds, nominal yields are lower for higher levels of real uncertainty. This reflects a fligh-to-quality effect, whereby bond prices (yields) respond positively (negatively) to macroeconomic uncertainty. For longer-term maturities, however, bond prices (yields) respond negatively (positively) to real uncertainty, consistent with the fact that assets dislike macroeconomic uncertainty.

It may be useful to compare some of our results more closely to those in Bansal and Shaliastovich (2012). In our framework, nominal yields respond negatively to inflation uncertainty at all maturities, while they respond negatively to real uncertainty at short horizons, and positively at longer horizons. Thus, the nominal uncertainty channel induces a fligh-to-quality effect, which dominates the intertemporal substitution motive at short horizongs for the real uncertainty channel. These implications are the mirror image of the results in Bansal and Shaliastovich (2012). We note that our implications are not directly comparable to theirs for several reasons. For one thing, they model real and nominal uncertainties, which affect respectively expected real and nominal growth. The conditional volatilities of realized consumption growth and inflation are, however, constant, while their are time-varying in our set-up. For another, they allow for non-neutrality of inflation. Hence expected inflation can negatively affect future growth. This specification is supported by a negatively estimated relationship in the data, although the coefficient is statistically insignificant. Our model features inflation-neutrality as in Wachter (2005) or Piazzesi and Schneider (2006), among many others, but we allow innovations in consumption growth to affect both realized and ex-

[23] Note that all three risk factors enter equation (23) in a highly non-linear way, and we cannot study the sensitivity of nominal bond yields to real uncertainty independent from expected price growth and inflation uncertainty.

pected inflation.[24] Understanding the differences in these modeling processes, and how they differentially affect the intertemporal substitution and precuationary savings effect through the nominal and real uncertainty channels, is an interesting question to pursue in future research.

To summarize, with GDA preferences, a simple economy with three sources of risk, expected inflation and real and inflation uncertainty, can generate an upward sloping term structure of nominal bond yields. Risk premia rise in response to shocks to expected inflation and decrease in response to shocks to inflation uncertainty. For short-term maturities, the flight-to-quality effect dominates for the response of nominal yields to a rise in real uncertainty. In contrast, the intertemporal substitution effect dominates for longer maturities, implying that longer-horizon nominal bond prices drop, and nominal yields rise, in response to higher macroeconomic uncertainty. We emphasize that we are able to generate an upward term structure of nominal bond yields without shocks to expected growth, nor do we need inflation non-neutrality, as in Bansal and Shaliastovich (2012). At the same time, we generate countercyclical real interest rates, in contrast to an upward sloping term structure of real interest rates in Wachter (2005). The success of the model is partly due to the ability of the GDA preferences to generate variation in the pricing kernel that is much larger than what we can obtain from standard recursive utility with symmetric preferences. To see this, note that we can use equation (5) to express the conditional variance of the log stochastic discount factor, $m_{t,t+1} = \ln M_{t,t+1}$, as

$$\begin{aligned} Var_t[m_{t,t+1}] = Var_t[m^*_{t,t+1}] + (\ln(1+\ell))^2\, Var_t[I\,(V_{t+1} < \kappa \mathcal{R}_t\,(V_{t+1}))] \\ + 2\ln(1+\ell)\, Cov_t[m^*_{t,t+1}, I\,(V_{t+1} < \kappa \mathcal{R}_t\,(V_{t+1}))], \end{aligned} \tag{29}$$

where $Var_t[m^*_{t,t+1}]$ represents the conditional variance of the pricing kernel with Epstein and Zin (1989) recursive utility. The two additional terms are strictly non-negative. As

[24] We note that the working paper version of Bansal and Shaliastovich (2012) featured inflation neutrality and sensitivity of expected and realized inflation to innovation in consumption growth.

a consequence, the conditional variance of the log pricing kernel with GDA preferences is always at least as large as that of an equivalent pricing kernel without GDA. This feature is a useful ingredient to solve asset pricing puzzles such as the equity premium and risk-free rate puzzles of, respectively, Mehra and Prescott (1985) and Weil (1989).[25] Another merit of the framework with asymmetric preferences is that state probabilities are reshuffled if an outcome is below a fraction κ of the certainty equivalent. Through this mechanism, the model endogenously generates effective countercylical risk aversion. This property is useful to explain stylized predictability patterns. In the next section, we exploit the simulated time series of nominal bond yields to show that the model can also account for the failure of the expectations hypothesis.

5 Predictability

In addition to the application to the term structure of interest rates, we show how the model can rationalize the failure of the expecations hypothesis. The empirical failure of the expectations hypothesis of interest rates, documented by Fama and Bliss (1987) and Campbell and Shiller (1991), has motivated the development of economic models seeking to explain the economic mechanism at work. Backus et al. (1989) show that the standard Consumption Capital Asset Pricing Model (CCAPM) with power utility cannot account for the anomaly. Recent thoretical explanations that have been suggested use the long run risk framework of Bansal and Yaron (2004) or the external habit set up of Campbell and Cochrane (1999). Wachter (2006) uses external habits with countercyclical interest rates to generate an upward sloping term structure of nominal bonds and predictability in bond returns. An alternative set up with habit preferences has been suggested by Le et al. (2010). In contrast, Bansal and Shaliastovich (2012) suggest that time-varying expected growth rates, expected inflation, real and inflation uncertainty together with Epstein and

[25] See Bonomo et al. (2011) for an application of GDA preferences to the equity and risk-free rate puzzles.

Zin (1989) recursive preferences and preference for early resolution of uncertainty, can yield time-varying bond risk premia and an upward sloping term structure of interest rates.[26] Our solutions, which generate an upward-sloping term structure of nominal bonds, are also able to reproduce predictability pattern in bond returns that are quantitatively close to the standard tests of the expectation hypothesis. We note that consumption growth predictability is not needed for our results.

5.1 The Expectations Hypothesis

The expectations hypothesis predicts that excess returns on bonds are unpredictable and that risk premia are constant. Various versions of the theory have been empirically tested in the literature and all yield the same conclusion that there is significant evidence of predictability in bond returns and that risk premia are time-varying. The power of the model can be evaluated by its ability to reproduce the multiplicity of empirical regression results suggested over the last decade. To be more specific, Fama and Bliss (1987) project holding period returns on the corresponding forward-spot spread[27]

$$rx_{t+12}^{(n)} = \alpha^{(n)} + \beta^{(n)} \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+12}^{(n)}, \tag{30}$$

where $rx_{t+12}^{(n)}$ indicates the annual excess log return of a n-year bond over the one-year yield $y_t^{(1)}$ defined as

$$rx_{t+12}^{(n)} = r_{t+12}^{(n)} - y_t^{(1)}, \tag{31}$$

[26] While these are the most recent articles on the topic, other relevant references are Bekaert et al. (1997), Longstaff (2000), Bekaert and Hodrick (2001), Bansal and Zhou (2002), Dai and Singleton (2002b) and Buraschi and Jiltsov (2005).

[27] We emphasize that all t subscripts refer to a monthly sampling frequency, while all n superscripts refer to the bond maturity in years.

with the return given by the difference in log prices, that is

$$r_{t+12}^{(n)} = p_{t+12}^{(n-1)} - p_t^{(n)}. \tag{32}$$

Alternatively, Campbell and Shiller (1991) regress changes in long yields on the yield spread

$$y_{t+12}^{(n-1)} - y_t^{(n)} = \alpha^{(n)} + \beta^{(n)} \frac{1}{n-1} \left(y_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+12}^{(n)}. \tag{33}$$

Another version of this is to test predictability of changes in short rates by the forward-spot spread

$$y_{t+12}^{(1)} - y_t^{(1)} = \alpha^{(n)} + \beta^{(n)} \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+12}^{(n)}, \tag{34}$$

where the forward spread $f_t^{(n)}$ for a loan between time $t + 12(n-1)$ and time $t + 12n$ is defined as

$$f_t^{(n)} = p_t^{(n-1)} - p_t^{(n)}. \tag{35}$$

All these regressions predict a slope coefficient of one. This outcome is strongly rejected in the data. Dai and Singleton (2002a) suggest that adding the bond risk premium to the left-hand side of the regression can restore the unity regression coefficient. Thus a model able to bring the slope coefficient closer to its predicted value should help resolving the expectations hypothesis puzzle. On that account, we also test our model on their adjusted regression given by

$$y_{t+12}^{(n-1)} - y_t^{(n)} + \frac{1}{n-1} \hat{E}_t \left[r_{t+12}^{(n)} - y_t^{(1)} \right] = \alpha^{(n)} + \beta^{(n)} \frac{1}{n-1} \left(y_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+12}^{(n)}, \tag{36}$$

where $\hat{E}_t \left[r_{t+12}^{(n)} - y_t^{(1)} \right]$ defines the risk premium component on nominal bond yields. Our last test is based on the results of Cochrane and Piazzesi (2005), who show that a single factor projection based on one to five-year forward rates captures a significant variation in bond returns. The single tent-shaped factor is obtained in a two-step estimation procedure, whereby first the average of one-year excess bond returns of two to five years to maturity are regressed on one to five year forward rates

$$\frac{1}{4}\sum_{n=2}^{5} rx_{t+12}^{(n)} = \gamma_0^{(n)} + \gamma_1^{(n)} y_t^{(1)} + \gamma_2^{(n)} f_t^{(2)} + ... + \gamma_5^{(n)} f_t^{(5)} + \varepsilon_{t+12}^{(n)}. \tag{37}$$

The predicted single bond factor $CP_t = \hat{rx}_{t+12}$ is then used in a second step to forecast excess bond returns at each maturity from two to five years

$$rx_{t+12}^{(n)} = \beta^{(n)} CP_t + \varepsilon_{t+12}^{(n)}. \tag{38}$$

5.2 Results

We first show that the model is able to generate the tent-shaped pattern of the Cochrane-Piazzesi factor. This is illustrated graphically in Figure 3. Panel A in Table 4 reports the unrestricted beta coefficients from the projection of excess returns on all forward rates. While we do replicate the shape of the loadings on forward rates, we emphasize that their sizes are an order of magnitude larger. The differences between the model and the data substantially shrink if we regress the average excess return on all forward rates. These results are reported in Panel B. The model-implied coefficient for the one-year forward rate of -1.29 compares favoribly to -1.39 in the data. The value of the coefficients peaks at the three-year forward rate with a value of 3.41, compared to an empirical estimate of 2.07. The values then fall down again to -0.65 in the model relative to -1.26 in the data. The model-implied R^2 of 23% is very close to the explanatory power in sample of 24%. Overall, these patterns already show that the model qualitatively replicates the tent-shaped pattern for the unrestricted regressions, and quantitatively for the regression of average excess returns on forward rates.

Finally, we report in Panel C of Table 4 the restricted single factor regression, and we plot the factor loadings in the lower panel of Figure 3a. The close match between both the regression coefficients and the R^2 of the regressions is striking. The slope coefficients range from 0.39 to 1.61 from the two- to five-year maturity in the model, and from 0.46 to 1.44 in the data. The model-implied standard errors are significantly smaller, which arises from the fact that we compare a population of 300,000 observations to an observed sample that is comparatively small. In addition, over the sample period 1964 to 2011, the single factor forecasts excess bond returns with an R^2 statistic of 21% for bonds with a maturity of two years. Predictability increases up to 4 years with an R^2 of 26% and flattens out a bit with a value of 24% for 5-year bonds. The model statistics are roughly in the same area, ranging from 26% at the 2-year horizon to 21% at the 5-year maturity.

In Table 5, we report the model-implied results of the Fama and Bliss (1987), the regressions of changes in short rates on forward-spot spreads, the Campbell and Shiller (1991) and the Dai and Singleton (2002a) regressions. We compare all model-implied results to the empirical regressions, estimated in the data using annual prices from 1964 to 2011. Overall, we confirm previous empirical evidence of predictabiliy of excess bond returns by bond yields. The implied statistics are quantitatively mostly comparable to the data counterparts, both in terms of regression coefficients and in terms of explanatory power, except for the short rate regressions in Panel B. We emphasize that for the adjusted Dai-Singleton regression, the beta coefficients come surprisingly close to one. This suggests that the model is able to generate a sizable time-varying risk premium in bond returns that is able to explain the failure of the expectations hypothesis for nominal bonds.

5.3 Conclusion

This paper proposes a solution method to models with generalized disappointment averse preferences when the economy is modeled to have continuous states, consistent with the observed data. In order to obtain accurate solutions, we limit the solution techniques to one-dimensional integral calculation. This is feasible if we specify the endowment economy to depend on a single state variable, macroeconomic uncertainty, which inhibits the same shocks as realized aggregate consumption growth.

We apply the framework to the term structure of nominal interest rates. We choose non-predictable consumption growth with time-varying uncertainty, specified as affine GARCH volatility dynamics. Likewise, we define affine GARCH dynamics for inflation uncertainty, which impacts both realized and excpected inflation. The key ingredients to the model preferences are (generalized) disappointment aversion and preferences for early resolution of uncertainty. Real bonds depend only on real uncertainty, while nominal bonds also depend on expected inflation and nominal uncertainty. The ability of the model to generate strong countercylical risk aversion and high volatility of the pricing kernel enable us to generate an upward term structure of nominal bond yields and volatilities consistent with the data. The model is also able to account for the failure of the expectations hypothesis. We are able to generate predictability in excess bond returns with time variation in risk premia that is closely tied to the sample estimates of most standard tests of the expectations hypothesis.

References

Aizenman, J. and Hausmann, R. (1994). Why is inflation skewed? a debt and volatility story, *NBER Working Papers 4837*, National Bureau of Economic Research, Inc.

Augustin, P. and Tédongap, R. (2014). Real economic shocks and sovereign credit risk, *Forthcoming, Journal of Financial and Quantitative Analysis* .

Backus, D. K., Gregory, A. W. and Zin, S. E. (1989). Risk premiums in the term structure: Evidence from artificial economies, *Journal of Monetary Economics* **24**(3): 371 – 399.

Bansal, R., Khatchatrian, V. and Yaron, A. (2005). Interpretable asset markets?, *European Economic Review* **49**(3): 531 – 560.

Bansal, R., Kiku, D. and Yaron, A. (2012). An empirical evaluation of the long-run risks model for asset prices, *Critical Finance Review* **1**(1): 183–221.

Bansal, R. and Shaliastovich, I. (2012). A long-run risks explanation of predictability puzzles in bond and currency markets, *Review of Financial Studies* .

Bansal, R. and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles, *The Journal of Finance* **59**(4): 1481–1509.

Bansal, R. and Zhou, H. (2002). Term structure of interest rates with regime shifts, *The Journal of Finance* **57**(5): 1997–2043.

Bekaert, G. and Hodrick, R. J. (2001). Expectations hypotheses tests, *The Journal of Finance* **56**(4): 1357–1394.

Bekaert, G., Hodrick, R. J. and Marshall, D. A. (1997). On biases in tests of the expectations hypothesis of the term structure of interest rates, *Journal of Financial Economics* **44**(3): 309 – 348.

Bonomo, M., Garcia, R., Meddahi, N. and Tédongap, R. (2011). Generalized disappointment aversion, long-run volatility risk and aggregate asset prices, *The Review of Financial Studies* **24**(1): 82–122.

Buraschi, A. and Jiltsov, A. (2005). Inflation risk premia and the expectations hypothesis, *Journal of Financial Economics* **75**(2): 429 – 490.

Campanale, C., Castro, R. L. d. and Clementi, G. L. (2010). Asset pricing in a production economy with chew-dekel preferences, *Review of Economic Dynamics* **13**(2): 379–402.

Campbell, J. Y. and Cochrane, J. H. (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* **107**(2): pp. 205–251.

Campbell, J. Y. and Shiller, R. J. (1991). Yield spreads and interest rate movements: A bird's eye view, *The Review of Economic Studies* **58**(3): pp. 495–514.

Campbell, J. Y., Shiller, R. J. and Viceira, L. M. (2009). Understanding inflation-indexed bond markets, *Brookings Papers on Economic Activity* pp. 79–120.

Chaudhuri, K., Greenwood-Nimmo, M., Kim, M. and Shin, Y. (2013). On the asymmetric u-shaped relationship between inflation, inflation uncertainty, and relative price skewness in the uk, *Journal of Money, Credit and Banking* **45**(7): 1431–1449.

Cochrane, J. H. and Piazzesi, M. (2005). Bond risk premia, *The American Economic Review* **95**(1): pp. 138–160.

Dai, Q. and Singleton, K. J. (2002a). Expectation puzzles, time-varying risk premia, and affine models of the term structure, *Journal of Financial Economics* **63**(3): 415 – 441.

Dai, Q. and Singleton, K. J. (2002b). Expectation puzzles, time-varying risk premia, and affine models of the term structure, *Journal of Financial Economics* **63**(3): 415 – 441.

Delikouras, S. (2014). Why corporate bonds may disappoint: Disappointment aversion and the credit spread puzzle, *Working Paper* .

Dolmas, J. (2013). Disastrous disappointments: Asset-pricing with disaster risk and disappointment aversion, *Working Paper Federal Reserve Bank of Dallas* .

Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* **57**(4): 937–969.

Evans, M. D. D. (1998). Real rates, expected inflation, and inflation risk premia, *The Journal of Finance* **53**(1): pp. 187–218.

Fama, E. F. and Bliss, R. R. (1987). The information in long-maturity forward rates., *American Economic Review* **77**(4): 680.

Farago, A. and Tédongap, R. (2014). Volatility downside risk, *Working Paper Stockholm School of Economics* .

Gul, F. (1991). A theory of disappointment aversion, *Econometrica* **59**(3): 667–86.

Hansen, L. P., Heaton, J., Lee, J. and Roussanov, N. (2007). Chapter 61 intertemporal substitution and risk aversion, Vol. 6, Part A of *Handbook of Econometrics*, Elsevier, pp. 3967 – 4056.

Heston, S. and Nandi, S. (2000). A Closed-Form GARCH Option Valuation Model, *Review of Financial Studies* **13** (3): 585–625.

Kandel, S. and Stambaugh, R. F. (1990). Expectations and volatility of consumption and asset returns, *The Review of Financial Studies* **3**: 207–232.

Kreps, D. M. and Porteus, E. L. (1978). Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* **46**(1): 185–200.

Le, A. and Singleton, K. J. (2013). The structure of risks in equilibrium affine models of bond yields, *Working Paper* .

Le, A., Singleton, K. J. and Dai, Q. (2010). Discrete-time affineq term structure models with generalized market prices of risk, *Review of Financial Studies* **23**(5): 2184–2227.

Lettau, M., Ludvigson, S. and Wachter, J. (2006). The declining equity premium: What role does macroeconomic risk play?, *Review of Financial Studies* .

Longstaff, F. A. (2000). The term structure of very short-term rates: New evidence for the expectations hypothesis, *Journal of Financial Economics* **58**(3): 397 – 415.

Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle, *Journal of Monetary Economics* **15**(2): 145 – 161.

Nelson, D. B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* **59** (2): 347–70.

Piazzesi, M. and Schneider, M. (2006). Equilibrium yield curves, *Working Paper 12609*, National Bureau of Economic Research.

Routledge, B. R. and Zin, S. E. (2010). Generalized disappointment aversion and asset prices, *Journal of Finance* **65**(4): 1303–1332.

Smyth, G. K. (1998). *Numerical Integration, in Encyclopedia of Biostatistics*, Vol. ISBN 0471 975761, John Wiley & Sons.

Stock, J. and Watson, M. (2002). Has the business cycle changed and why?, *in M. Gertler and Kenneth Rogoff, NBER Macroeconomics Annual: 2002, MIT Press Cambridge* .

Tédongap, R. (2014). Consumption volatility and the cross-section of stock returns, *forthcoming in The Review of Finance* .

Timmermann, A. (2000). Moments of markov switching models, *Journal of Econometrics* **96**(1): 75 – 111.

Wachter, J. A. (2005). Solving models with external habit, *Finance Research Letters* **2**(4): 210 – 226.

Wachter, J. A. (2006). A consumption-based model of the term structure of interest rates, *Journal of Financial Economics* **79**(2): 365 – 399.

Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle, *Journal of Monetary Economics* **24**(3): 401 – 421.

Table 1: Model Parameter Calibration

This table reports model and preference parameter values, which are calibrated at a monthly decision interval.

Parameter		Value
Consumption Growth Dynamics		
Mean consumption growth	μ_c	0.0015
Persistence of volatility	ϕ_σ	0.9950
Volatility level	$\sqrt{\mu_\sigma}$	0.7305e-02
Volatility of volatility	σ_σ	0.6263e-04
Inflation Dynamics		
Mean inflation rate	μ_π	0.0030
Persistence of expected inflation	ϕ_z	0.9840
Inflation leverage on news	ν_π	-0.1294
Level of expected inflation shock volatility	ν_z	0.3457
Inflation Volatility level	μ_v	6.3698e-07
Persistence of inflation volatility	ϕ_v	0.8500
Level of residual inflation volatility	ν_v	9.5546e-08
Inflation leverage coefficient	β_v	-2.9827e+03
Preference Parameter Values		
Subjective discount factor	δ	0.9989
Intertemporal elasticity of substitution	ψ	1.5
Coefficient of relative risk aversion	γ	2.5
Coefficient of disappointment aversion	ℓ	1
Coefficient of generalized disappointment aversion	κ	1

Table 2: Cash-flows

This table presents moments of consumption and inflation dynamics from the data and the model. The data are real, sampled at an annual frequency, and cover the period 1929 to 2011. Standard errors are Newey-West with one lag. For the model, we report population statistics based on a simulation of 300,000 months. Consumption and price growth rates in the model are time-averaged. Data for consumption and price growth are taken from the Bureau of Economic Analysis National Income and Product Accounts Tables.

Consumption				Inflation			
Moment	Estimate	SE	Model	Moment	Estimate	SE	Model
$E[\Delta_c]$ (%)	1.97	0.28	1.79	$E[\pi]$ (%)	3.17	0.52	3.57
$\sigma[\Delta_c]$ (%)	2.02	0.38	2.08	$\sigma[\pi]$ (%)	3.29	0.76	2.89
$AC1[\Delta_c]$	0.48	0.12	0.24	$AC1[\pi]$	0.83	0.13	0.86
$Skew[\Delta_c]$	-1.58	0.67	-0.70	$Skew[\pi]$	-0.80	1.23	1.34
$Kurt[\Delta_c]$	9.63	2.29	7.88	$Kurt[\pi]$	8.64	2.13	6.15

Table 3: Asset Pricing Implications in Population

This table reports the term structure of nominal interest rates and the corresponding volatilities. All asset pricing implications in population are based on simulations of 300,000 months of data. Data statistics are based on the Fama-Bliss zero-coupon database from CRSP over the sample period 1964 until 2011.

Nominal Term structure of Interest Rates - Model					
	1y	2y	3y	4y	5y
Mean Yield (%)	4.72	5.20	5.72	6.27	6.85
Std (%)	2.25	2.23	2.24	2.28	2.33

Nominal Term structure of Interest Rates - Data					
	1y	2y	3y	4y	5y
Mean Yield (%)	5.20	5.40	5.58	5.73	5.83
Std (%)	3.03	2.97	2.90	2.85	2.78

Table 4: Cochrane-Piazzesi Regressions

Panel A reports the regression results from excess bond returns on all forward rates. Panel B reports the regression results from the projection of average excess return on all forward rates. Panel C reports the single factor regressions. The model-implied regressions are based on a simulated sample of 300,000 months of data. The estimated results from the data are based on Fama-Bliss discount bond prices from 1964 to 2011.

Panel A: Regressions of excess returns on all forward rates
$rx_{t+12}^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_2^{(n)} f_t^{(2)} + ... + \beta_5^{(n)} f_t^{(5)} + \varepsilon_{t+12}^{(n)}$

	Model							**Data**						
Statistic	β_0	β_1	β_2	β_3	β_4	β_5	R^2	β_0	β_1	β_2	β_3	β_4	β_5	R^2
OLS,n=2	-0.10	-0.48	-0.01	-1.50	5.21	-3.04	0.27	-1.04	-0.64	0.01	0.84	0.49	-0.53	0.22
HH,12 lags	0.56	0.17	0.68	1.47	2.28	1.30	NaN	0.70	0.21	0.36	0.37	0.18	0.23	NaN
NW,18 lags	0.50	0.16	0.62	1.37	2.09	1.19	NaN	0.62	0.20	0.35	0.36	0.20	0.24	NaN
OLS,n=3	-0.34	-0.91	-0.72	0.61	4.81	-3.44	0.25	-1.49	-1.14	-0.56	2.30	0.78	-1.16	0.23
HH,12 lags	1.07	0.34	1.33	2.85	4.38	2.51	NaN	1.31	0.40	0.61	0.65	0.36	0.41	NaN
NW,18 lags	0.97	0.31	1.22	2.65	4.02	2.28	NaN	1.16	0.38	0.61	0.63	0.38	0.43	NaN
OLS,n=4	-1.24	-1.51	-0.72	4.27	-0.68	-0.85	0.23	-2.07	-1.68	-0.63	2.48	1.87	-1.76	0.26
HH,12 lags	1.56	0.50	1.97	4.20	6.37	3.64	NaN	1.81	0.57	0.80	0.87	0.51	0.57	NaN
NW,18 lags	1.41	0.46	1.82	3.90	5.85	3.32	NaN	1.61	0.53	0.80	0.84	0.53	0.59	NaN
OLS,n=5	-2.75	-2.27	-0.38	10.27	-11.71	4.75	0.21	-2.72	-2.11	-0.62	2.64	2.02	-1.60	0.24
HH,12 lags	2.03	0.66	2.62	5.54	8.29	4.73	NaN	2.24	0.70	0.95	1.04	0.63	0.70	NaN
NW,18 lags	1.84	0.61	2.41	5.15	7.62	4.31	NaN	1.99	0.65	0.97	1.02	0.67	0.73	NaN

Panel B: Regression of average excess return on all forward rates
$\overline{rx}_{t+12} = \gamma^\top f_t + \bar{\varepsilon}_{t+12}$

Statistic	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2
OLS	-1.11	-1.29	-0.46	3.41	-0.59	-0.65	0.23	-1.83	-1.39	-0.45	2.07	1.29	-1.26	0.24
HH,12 lags	1.30	0.42	1.64	3.50	5.31	3.04	NaN	1.51	0.47	0.67	0.73	0.42	0.48	NaN
NW,18 lags	1.18	0.38	1.51	3.25	4.88	2.77	NaN	1.34	0.44	0.67	0.71	0.44	0.50	NaN

Panel C: Restricted single factor regressions
$rx_{t+12}^{(n)} = b_n \left(\gamma^\top f_t\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
b_n	0.39	0.79	1.20	1.61	b_n	0.46	0.86	1.25	1.44
HH,12 lags	0.00	0.01	0.01	0.01	HH,12 lags	0.08	0.16	0.22	0.27
R^2	0.26	0.25	0.23	0.21	R^2	0.21	0.23	0.26	0.24

Table 5: Benchmark Regressions

Panel A reports the restricted Cochrane-Piazzesi regressions from the projection of holding period returns on the single CP factor, Panel B reports the Fama-Bliss regression results from the projection of holding period returns on forward-spot spreads. Panel C reports the short-rate regression results from the projection of changes in short rates on forward-spot spreads. Panel D reports the Campbell-Shiller regressions from the projection of changes in long rate spreads on yield-spot spreads. Panel E reports the Dai-Singleton regressions from the projection of adjusted changes in long rate spreads on yield-spot spread.

Model | **Data**

Panel A: Cochrane-Piazzesi: regression of holding period returns on single CP factor
$rx_{t+12}^{(n)} = b_n\left(\gamma^\top f_t\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
b_n	0.39	0.79	1.20	1.61	b_n	0.46	0.86	1.25	1.44
HH,12 lags	0.00	0.01	0.01	0.01	HH,12 lags	0.08	0.16	0.22	0.27
R^2	0.26	0.25	0.23	0.21	R^2	0.21	0.23	0.26	0.24

Panel B: Fama-Bliss: regression of holding period returns on forward-spot spread
$rx_{t+12}^{(n)} = \alpha^{(n)} + \beta^{(n)}\left(f_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
β_n	0.98	1.02	1.05	1.08	β_n	0.83	1.14	1.38	1.10
HH,12 lags	0.02	0.02	0.02	0.02	HH,12 lags	0.26	0.35	0.41	0.46
NW,18 lags	0.01	0.02	0.02	0.02	NW,18 lags	0.23	0.31	0.36	0.42
R^2	0.19	0.17	0.16	0.14	R^2	0.12	0.13	0.15	0.08

Panel C: Short Rates: regression of changes in short rate on forward-spot spread
$y_{t+12}^{(1)} - y_t^{(1)} = \alpha^{(n)} + \beta^{(n)}\left(f_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
β_n	0.02	0.02	0.03	0.04	β_n	0.17	0.52	0.82	0.96
HH,12 lags	0.02	0.02	0.02	0.02	HH,12 lags	0.26	0.32	0.26	0.18
NW,18 lags	0.01	0.01	0.01	0.01	NW,18 lags	0.23	0.33	0.27	0.23
R^2	0.00	0.00	0.00	0.00	R^2	0.01	0.05	0.13	0.18

Panel D: Campbell-Shiller: regression of changes in long rate spreads on yield-spot spread
$y_{t+12}^{(n-1)} - y_t^{(n)} = \alpha^{(n)} + \beta^{(n)}\frac{1}{n-1}\left(y_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
β_n	-0.97	-0.96	-0.95	-0.95	β_n	-0.67	-1.07	-1.47	-1.48
HH,12 lags	0.03	0.03	0.03	0.04	HH,12 lags	0.53	0.64	0.70	0.74
NW,18 lags	0.03	0.03	0.03	0.03	NW,18 lags	0.47	0.56	0.62	0.66
R^2	0.05	0.05	0.04	0.04	R^2	0.02	0.04	0.06	0.06

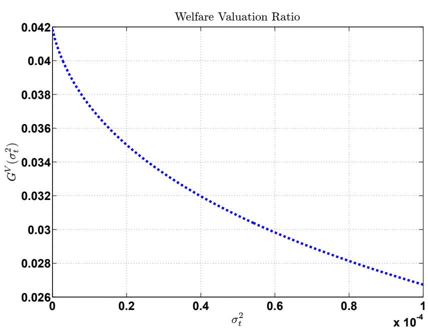
Panel E: Dai-Singleton: regression of adjusted changes in long rate spreads on yield-spot spread
$y_{t+12}^{(n-1)} - y_t^{(n)} + \frac{1}{n-1}\hat{E}_t\left[r_{t+12}^{(n)} - y_t^{(1)}\right] = \alpha^{(n)} + \beta^{(n)}\frac{1}{n-1}\left(y_t^{(n)} - y_t^{(1)}\right) + \varepsilon_{t+12}^{(n)}$

	n=2	n=3	n=4	n=5		n=2	n=3	n=4	n=5
β_n	0.70	0.86	1.01	1.16	β_n	1.10	1.13	1.02	0.74
HH,12 lags	0.03	0.03	0.03	0.04	HH,12 lags	0.46	0.57	0.63	0.66
NW,18 lags	0.03	0.03	0.03	0.03	NW,18 lags	0.42	0.51	0.56	0.59
R^2	0.03	0.04	0.05	0.06	R^2	0.06	0.05	0.04	0.02

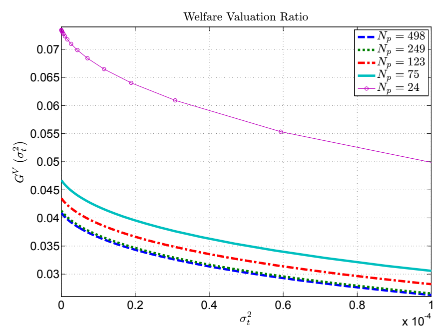
Figure 1: Model Solutions

Figure 1a plots the welfare valuation ratio $V_t/C_t = G^V(\sigma_t^2)$ as a function of consumption volatility σ_t^2 for our benchmark scenario with 501 grid points. Figure 1b plots the welfare valuation ratio $V_t/C_t = G^V(\sigma_t^2)$ as a function of consumption volatility σ_t^2 for different grids, ranging from 24 to 498 points. Figure 1c reports the probability of disappointment $\xi(\sigma_t^2)$ as a function of consumption volatility σ_t^2. Figure 1d plots the real yields $y_t^{(n)}$ as a function of consumption volatility for maturities $n = 1$ year to $n = 5$ years. Figures 1e and 1f plot the sensitivities of nominal bond yields $y_t^{(n)}$ to, respectively, expected inflation $(-B^{\$}_{z,n}/n)$ and inflation volatility $(-B^{\$}_{v,n}/n)$.

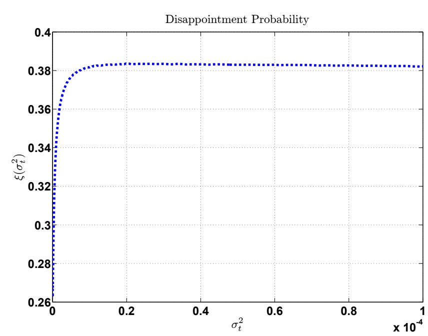
(a)



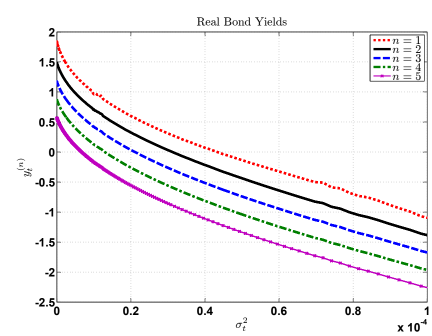
(b)



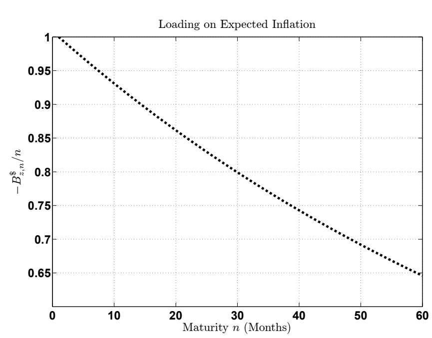
(c)



(d)



(e)



(f)

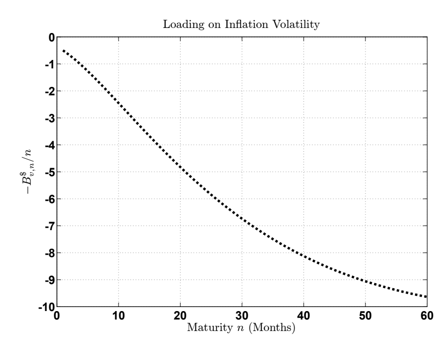


Figure 2: Nominal Bond Yields

This figure plots nominal bond yields $y_t^{\$,(n)}$ as a function of consumption volatility for maturities $n = 1$ year to $n = 5$ years when expected inflation and inflation uncertainty are fixed at their long-run values.

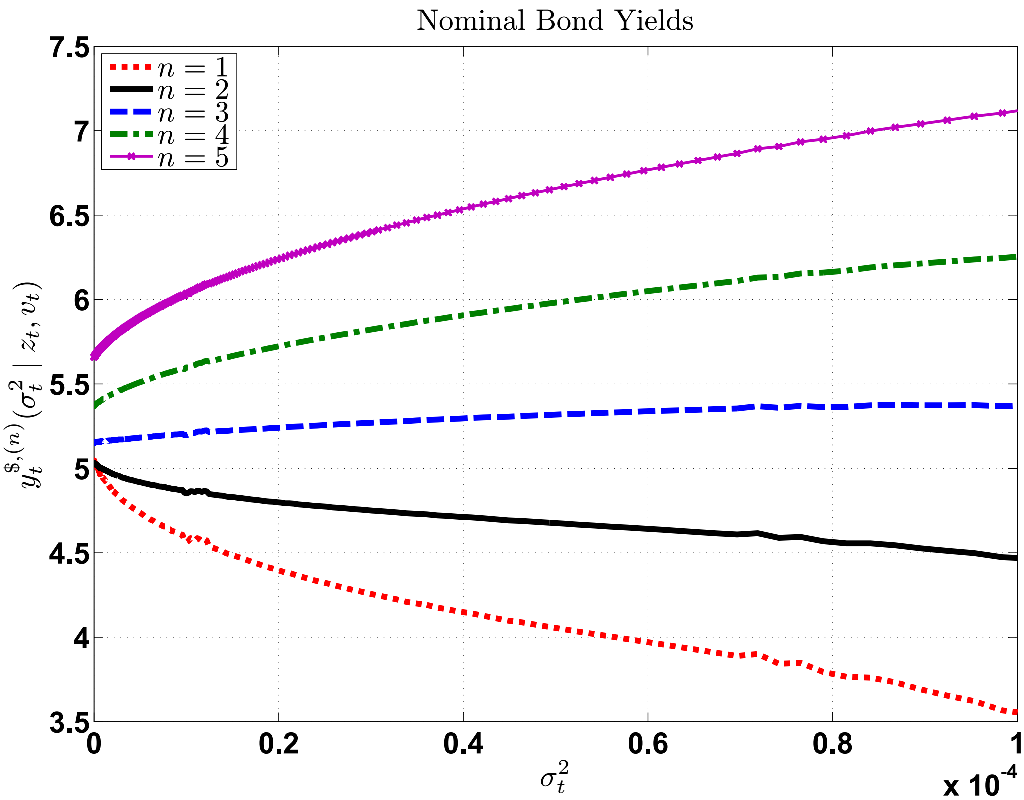
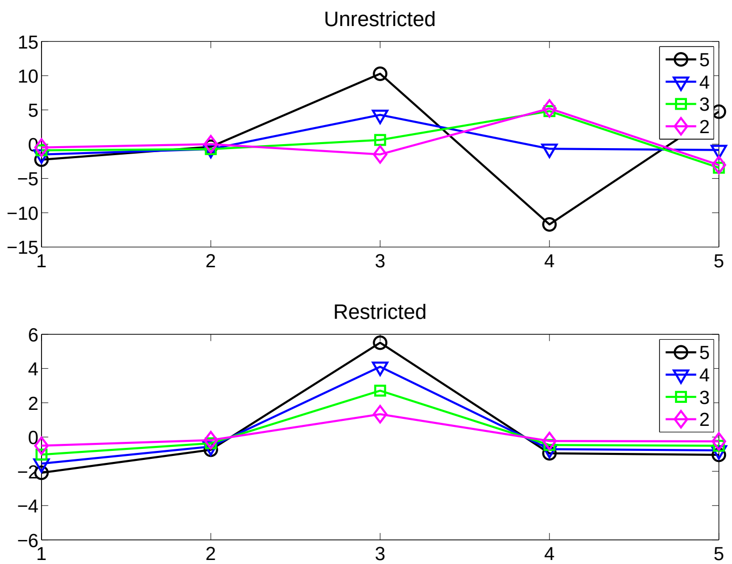


Figure 3: Regression Coefficients of one-year excess retruns on forward rates

This table plots the regression coefficients from the projection of excess bond returns (upper panel) and average excess returns (lower panel) on forward rates. The unrestricted model that projects excess returns on forward rates is specified as $rx_{t+12}^{(n)} = \beta_0^{(n)} + \beta_1^{(n)} y_t^{(1)} + \beta_2^{(n)} f_t^{(2)} + ... + \beta_5^{(n)} f_t^{(5)} + \varepsilon_{t+12}^{(n)}$. The restricted regression uses the single factor extracted from the projection of average excess returns on forward rates, specified as $\overline{rx}_{t+12} = \gamma^\top f_t + \bar{\varepsilon}_{t+12}$. The single factor is used in the restriced regression as $rx_{t+12}^{(n)} = \beta^{(n)} \left(\gamma^\top f_t\right) + \varepsilon_{t+12}^{(n)}$. The left panel reports results for the model-implied regressions based on simulated data of 300,000 months. The right panel reports the estimated results from the data using Fama-Bliss bond prices from 1964 to 2011.

Model

(a)



Data

(b)