Asymmetries and Portfolio Choice

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Abstract

We examine the portfolio choice of an investor with generalized disappointment aversion preferences who faces returns described by a normal-exponential model. We derive a three-fund separation strategy: the investor allocates wealth to a risk-free asset, a standard mean-variance efficient fund, and an additional fund reflecting return asymmetries. The optimal portfolio is characterized by the investor’s endogenous effective risk aversion and implicit asymmetry aversion. We find that disappointment aversion is associated with much larger asymmetry aversion than are standard preferences. Our model explains patterns in popular portfolio advice and provides a reason for shifting from bonds to stocks as the investment horizon increases.

Keywords: Asset allocation, downside risk.

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1 Introduction

Asset returns are asymmetrically distributed and display fatter tails than if they were normally distributed. Correlations between asset returns conditional on downside and upside moves display asymmetric patterns. In particular, correlations between stocks tend to be greater for downside than upside moves (see, e.g., Ang and Chen, 2002; and Hong et al., 2006), while long-term bonds tend to be negatively correlated with stocks conditional on down markets and positively correlated with stocks conditional on up markets (see, e.g., Baele et al., 2010; Campbell et al., 2013; and David and Veronesi, 2013). There is also evidence that investors have asymmetric attitudes towards risk across downward and upward movements. In particular, they place larger weights on losses than on gains when assessing the risk of their portfolio. This observation has led to utility theories that emphasize investors’ aversion to downside risk.

We study the joint impact of these two types of asymmetries, i.e., in asset returns and in investor attitudes towards risk, on investor portfolio choice. In particular, we investigate whether taking into account these asymmetries helps us understand patterns in popular portfolio advice that are puzzling to standard models. According to the two-fund separation theorem of standard portfolio theory, everyone should hold risky assets in the same proportion, and only the relative weights in the risky portfolio and in cash should vary across investors (Tobin, 1958). This is in sharp contrast with the portfolio recommendations of financial advisors, both across investors with different risk tolerances and with different investment horizons. First, regarding differences across risk tolerances, Canner et al. (1997) document that when dividing the portfolio between cash, bonds, and stocks, financial advisors often recommend that conservative investors should allocate more of their risky portfolio (i.e., bonds plus stocks) to bonds, while aggressive investors should allocate more to stocks. Canner et al. (1997) refer to the inconsistency of this advice with the two-fund separation result as the “asset allocation puzzle.” Second, regarding differences across investment hori-
zons, advisors typically recommend that investors with a short horizon should favor bonds in their risky portfolios, while long horizon investors should favor stocks. We argue in this paper that by extending the standard model to jointly take into account return and preference asymmetries, we can rationalize the above patterns in portfolio advice.

We propose a simple and parsimonious theoretical setup in a static setting, explicitly ruling out any effect that might otherwise arise from dynamic channels. In modeling asymmetric investor preferences, we rely on the disappointment aversion utility of Gul (1991) and its generalization by Routledge and Zin (2010). These preferences are consistent with the puzzling experimental behavior observed in the Allais (1979) paradox and provide a framework in which investors place different weights on downside losses and upside gains. Moreover, this utility specification is axiomatic, normative, and firmly grounded in formal decision theory under uncertainty. Choi et al. (2007) demonstrate that the disappointment aversion utility provides a good interpretation of individual-level data from a series of experiments of subjects facing a portfolio choice problem. Gill and Prowse (2012) provide experimental evidence that people are disappointment averse when they compete. A further advantage of these preferences is that power utility arises as a special case when the degree of disappointment aversion is zero. Our setting is therefore also convenient for studying optimal portfolios when only return asymmetries are present.

To model asymmetries in asset returns we propose a normal-exponential model. The model assumes that idiosyncratic security risks follow a multivariate normal distribution, while skewness is generated by a single common factor that follows an exponential distribution, but upon which different securities have different loadings. We argue that the proposed model acceptably captures both the skewness of individual asset returns and the coskewness between assets. Moreover, the model can match other key statistical features of the data such as fat tails and asymmetric correlations. A further advantage of using this model is that the normal distribution is a simple special case, when an asset’s sensitivity to the common shock is set to zero. Our setting is therefore also convenient for studying optimal portfolios.
when only preference asymmetry is present.

We derive an analytical solution to the portfolio choice problem and demonstrate that it leads to a three-fund separation strategy. The first fund is a risk-free asset while the second is a standard mean-variance efficient fund. The third fund, whose composition is determined by the asymmetry of the risky asset returns, is labeled the “asymmetry-variance” fund. The fund takes a short position in negatively skewed assets and long position in assets with positive skewness. The weight an investor assigns to each fund depends primarily on her preference parameters. Using the analytical solution, we can characterize the effects of asymmetries in returns and preferences.

If there is no asymmetry in returns, they become jointly normal and the asymmetry-variance fund becomes redundant. The standard two-fund separation applies, as the investor chooses to invest in only the risk-free asset and the mean-variance efficient fund. In this case, the effective risk aversion of the investor is enough to describe the optimal portfolio. For a power-utility investor, the effective risk aversion equals the relative risk-aversion coefficient of the utility function. For a disappointment-averse investor, the effective risk aversion depends also on the optimal portfolio, making it endogenous to the optimal choice.\(^1\) We contribute to the literature by deriving the formula for effective risk aversion when the investor has disappointment-aversion preferences. The effective risk aversion provides a convenient way to compare different parameterizations of preferences. We demonstrate that several sets of parameters of the generalized disappointment-aversion preferences (which have two additional parameters compared with the standard power utility) lead to the same effective risk aversion.

If there is asymmetry in the return distribution, the investor allocates some of her wealth to the asymmetry-variance fund. Apart from effective risk aversion, an implicit aversion to asymmetric returns is needed to describe the optimal choice of a given investor. We

\(^1\)The finding that effective risk aversion is endogenous under disappointment aversion is consistent with the discussion presented by Routledge and Zin (2010) and Bonomo et al. (2011).
demonstrate that the asymmetry aversion implied by disappointment-aversion preferences can differ significantly from the values implied by the standard power utility.

Using a calibrated example involving bonds and stocks as risky assets, we illustrate several implications of the model. The asymmetry aversion implied by power utility is positive but low in magnitude. An investor who is not disappointment averse allocates only a small fraction of her wealth to the asymmetry-variance fund. Standard symmetric preferences thus imply that return asymmetry only marginally affects the composition of optimal portfolios. In other words, the investor behaves as if she has mean-variance preferences. This is consistent with Levy and Markowitz (1979) and Hlawitschka (1994) who argue that mean-variance utility provides a fairly good approximation of most standard expected utility functions.

For a disappointment-averse investor, the optimal choice strongly depends on the reference point distinguishing disappointing from non-disappointing outcomes. First, when the reference point equals the certainty equivalent of the investment, a sufficiently disappointment-averse investor does not hold risky securities and instead invests all her wealth in the risk-free asset. This is in line with the findings of Ang et al. (2005). To the contrary, a disappointment-averse investor whose reference point differs from the certainty equivalent always finds it optimal to hold risky assets.

Second, when the reference point is lower than the certainty equivalent, the implicit asymmetry aversion is positive and large in magnitude compared to standard preferences. Negative asymmetry in asset returns is associated with an increased probability of large losses. Therefore, an investor with a focus on avoiding these losses will reduce her investment in negatively skewed assets by taking a relatively large long position (compared to a power utility investor) in the asymmetry-variance fund. This induces a shift from the negatively skewed stocks towards the bonds. Disappointment aversion can cause significant changes in the optimal asset allocation.

Third, we examine what happens when the disappointment threshold is higher than the investor’s certainty equivalent. Routledge and Zin (2010) briefly discuss the possibility of
this case, but otherwise the literature has largely ignored this setting. Although the setting where the reference point is lower than the certainty equivalent is arguably more relevant for the majority of investors, there might be certain investors with a higher threshold. We demonstrate what this implies for investor behavior. A disappointment-averse investor with a high threshold needs a high portfolio return in order not to be disappointed. She prefers stocks over the bonds within her risky portfolio because of their high upside potential. Since stocks are negatively skewed, the implicit asymmetry aversion becomes negative, making it seem that the investor likes negative skewness. However, in this case, the driving force of the choice is not the negative skewness of stocks but their high upside potential.

Generalized disappointment aversion can thus implicitly generate preference for positive skewness, similar to cumulative prospect theory preferences as demonstrated by Barberis and Huang (2008). However, it can also implicitly generate preference for negative skewness. In fact, depending on her disappointment threshold, the investor’s primary objective is either to avoid large losses or to chase big gains. The investor’s position in the asymmetry-variance fund is meant to increase her holding in the asset with the highest contribution to the optimal portfolio’s ability to meet this objective. Depending on that asset’s skewness, this means taking either a long or a short position in the asymmetry-variance fund, which implicitly determines the sign of the preference for skewness.

Asymmetry of asset returns coupled with generalized disappointment aversion generates significant variation in optimal positions in the asymmetry-variance fund across investors with different preferences (risk appetites). If conservative investors are more disappointment averse, they should invest a higher fraction of their wealth in the asymmetry-variance fund, and consequently have higher bond/stock allocation ratios than aggressive investors. We show in a separate calibration using the data from Canner et al. (1997) that taking into account return- and preference asymmetries resolves the asset allocation puzzle.\footnote{Other potential explanations for this divergence between theory and practice have been proposed in the literature. Bajeux-Besnainou et al. (2001) explain the puzzle by assuming that the investor’s horizon may exceed the maturity of the cash asset. Shalit and Yitzhaki (2003) use conditional stochastic dominance.
To understand portfolio recommendations across different horizons, we also study the effect of an increasing investment horizon. Consider a disappointment-averse investor with a disappointment threshold lower than the certainty equivalent. At short investment horizons, the negative skewness deters the investor from holding stocks. However, assuming independent return dynamics, asset returns become more symmetric as the investment horizon increases. Consequently, investors with longer investment horizons hold relatively more stocks than bonds within the risky portfolio. Hence, we provide a reason for shifting from stocks to bonds as the horizon decreases that differs from the reason due to the effective mean reversion in prices (see, e.g., Campbell and Viceira, 2002, 2005) or non-tradable human capital (see, e.g., Jagannathan and Kocherlakota, 1996; and Cocco et al., 2005).

Our work relates to the literature on the effect of skewness on optimal portfolios. The investor’s asymmetry aversion in these studies is usually implied by a Taylor’s expansion of some standard utility function (as in Conine and Tamarkin, 1981; Jondeau and Rockinger, 2006; Guidolin and Timmermann, 2008; Martellini and Ziemann, 2010; or Ghysels et al., 2014) or is set exogenously to an ad hoc value (as in Mitton and Vorkink, 2007; or Harvey et al., 2010). Our approach is different. We consider a nonstandard utility specification (with power utility as a special case), but instead of directly specifying the investor’s asymmetry aversion, we are interested in its value implied by disappointment-averse preferences.

Das and Uppal (2004) examine portfolio choice with systemic risks; we consider similar return asymmetries, but treat the investor as having nonstandard preferences and explicitly caring about the downside risk underlying the return asymmetry. Similar to Das and Uppal (2004), we quantify the certainty equivalent cost of ignoring return asymmetry. Our work also relates to Ang et al. (2005), who consider portfolio choice under disappointment aversion and normally distributed asset returns. We extend their analysis in several directions. First, we consider the generalized disappointment aversion utility in which the reference point arguments to demonstrate that advisors, acting as agents for numerous clients, recommend portfolios that are not inefficient for all risk-averse investors. Campbell and Viceira (2001) rationalize the popular advice in the context of intertemporal asset allocation models with time-varying expected returns.
distinguishing disappointing from non-disappointing outcomes deviates from the certainty equivalent. Second, we study the effect of asymmetric return distributions supported by data. Third, we derive an analytical solution to the optimal portfolio problem and our proposed setup easily accommodates multiple risky assets.

We focus on the portfolio choice implications of skewness; another strand of the literature focuses on its asset pricing implications. For example, Kraus and Litzenberger (1976), Harvey and Siddique (2000), and Dittmar (2002) provide evidence that coskewness is a priced factor in the cross-section of stock returns. The common starting point of these studies is to take the third- or fourth-order Taylor’s expansion of the representative investor’s utility function and demonstrate that the resulting stochastic discount factor (SDF) is a nonlinear function of the return on aggregate wealth. However, these studies do not take a stand on what original utility function is consistent with the estimated price of risk. Dittmar (2002) finds that the SDF implied by the power utility is rejected in the data. Langlois (2013) proposes a setup where the return distribution incorporates systematic and idiosyncratic asymmetry components and derives the asset pricing implications of the model. He provides evidence that the asymmetry related factors from the model are priced in the returns of various asset classes, but he does not consider what investor preferences are consistent with the magnitudes of the prices of risk.

In the Online Appendix we derive the asset pricing implications of our dual asymmetries setting, and show that they are similar to Simaan (1993). In particular, we find a negative relation between asset skewness and expected return, consistent with Barberis and Huang (2008), Mitton and Vorkink (2007) and Boyer et al. (2010) on the overpricing of positively skewed securities. Empirical tests of these implications using a large cross-section of assets is out of the scope of the current work and constitute an interesting avenue for future research. Finally, Bernardo and Ledoit (2000) develop an asset pricing approach based on a gain-loss ratio. In our setting, we show that at optimum all assets have the same gain-loss ratio, which is determined by the disappointment aversion coefficient.
2 Theoretical setup

An investor with a generalized disappointment-aversion utility as in Routledge and Zin (2010) can allocate wealth between \( N \) risky securities denoted \( i = 1, 2, \ldots, N \) and a risk-free asset denoted \( i = f \). Similar to Ang and Bekaert (2002), Das and Uppal (2004), Ang et al. (2005), and Guidolin and Timmermann (2008), we consider a finite-horizon setup with utility defined over terminal wealth. Our model for asset returns is set in discrete time.

2.1 Investor attitude towards risk

Generalized disappointment-aversion (GDA) preferences capture the idea that investors care differently about downside losses than about upside gains. The investor’s objective is to maximize the utility of the certainty equivalent of the terminal wealth, \( W \). Following Routledge and Zin (2010), the certainty equivalent of the terminal wealth \( R(W) \) is implicitly defined by

\[
\theta U(R(W)) = E[U(W)] - \ell E[(U(\kappa R(W)) - U(W))I(W < \kappa R(W))],
\]

where \( I(\cdot) \) is an indicator function that equals 1 if the condition is met and 0 otherwise, and

\[
U(X) = \begin{cases} 
X^{1-\gamma} & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\
\frac{1}{1-\gamma} \ln X & \text{if } \gamma = 1.
\end{cases}
\]

The parameter \( \gamma > 0 \) measures the investor’s risk aversion. The parameter \( \ell \geq 0 \) is the investor’s degree of disappointment aversion and \( \kappa > 0 \) is the percentage of her certainty equivalent below which outcomes are considered disappointing. The parameter \( \theta \) is defined as \( \theta \equiv 1 - \ell (1^{1-\gamma} - 1)I(\kappa > 1) \) and allows us to capture both sides of noncentral disappointment in a single setting. Routledge and Zin (2010) point out that monotonicity imposes the restriction that \( \theta > 0 \).
If the investor’s degree of disappointment aversion is zero ($\ell = 0$), the definition of the certainty equivalent from (1) simplifies to

$$U (\mathcal{R}(W)) = E [U (W)] .$$

(3)

In this case, the investor has expected utility (EU) preferences with power utility. When $\ell > 0$, outcomes lower than $\kappa \mathcal{R}(W)$ receive an extra weight and lower the investor’s certainty equivalent relative to EU. As the objective is to maximize the certainty equivalent, a disappointment-averse investor would like to avoid outcomes below $\kappa \mathcal{R}(W)$. The penalty for disappointing outcomes increases with $\ell$, so this parameter modulates the importance of disappointment versus satisfaction and can be interpreted as the degree of disappointment aversion. Parameter $\kappa$ sets the threshold for disappointing outcomes relative to the certainty equivalent. The special case $\kappa = 1$ corresponds to the original disappointment-aversion (DA) preferences of Gul (1991). If $\kappa < 1$, the random future value is considered disappointing if it lies sufficiently below today’s certainty equivalent; if $\kappa > 1$, the random future value must be sufficiently far above the certainty equivalent to be considered not disappointing. We demonstrate that different values of $\kappa$ lead to diverse investor behavior. We refer to an investor for whom $\kappa = 1$ as a DA investor and to an investor for whom $\kappa \neq 1$ as a GDA investor.

Terminal wealth may be written as

$$W = W_0 \mathcal{R}_W ,$$

(4)

where $W_0$ is the initial wealth and $\mathcal{R}_W$ is the gross return on the investor’s portfolio over the investment horizon. Due to the homogeneity of the utility function (2), the following

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3Throughout the paper, when we refer to the EU investor, it corresponds to the case where $U (\cdot)$ is the power utility.
equality holds:  
\[ \mathcal{R}(W) = W_0 \mathcal{R}(R_W). \]  

(5)

Ultimately, the investor’s objective is simply to maximize the certainty equivalent of the portfolio gross return, \( \mathcal{R}(R_W) \), given by

\[ \theta U(\mathcal{R}) = E[U(R_W)] - \ell E[(U(\kappa \mathcal{R}) - U(R_W)) I(R_W < \kappa \mathcal{R})], \]  

where we have used the short-hand notation \( \mathcal{R} \) for \( \mathcal{R}(R_W) \). Maximizing the certainty equivalent leads to the same solution as does maximizing its logarithm, \( \eta \equiv \ln \mathcal{R} \). We demonstrate in Appendix A that the investor’s log certainty equivalent is implicitly given by

\[ \eta = \begin{cases} 
\frac{1}{1-\gamma} \ln E[\exp((1-\gamma)r_W)] & \text{if } \gamma > 0 \text{ and } \gamma \neq 1 \\
-\frac{1}{1-\gamma} \ln (\theta + \ell \kappa^{1-\gamma} (1 - E[\exp((\gamma - 1)p_W)])) & \text{if } \gamma = 1 \end{cases}, \]

(7)

where

\[ p_W \equiv \max(\ln \kappa + \eta - r_W, 0). \]

(8)

corresponds to the payoff of a European put option on the portfolio’s log return \( r_W \), with a strike equal to \( \ln \kappa + \eta \), the investor’s endogenous threshold of disappointment.

The intuition for (7) is most straightforward when \( \gamma = 1 \). The investor’s log certainty equivalent is a sum of two components. The first component is the log certainty equivalent of the EU investor, while the second component is a downside risk penalty for achieving a portfolio return below the endogenous disappointment threshold. The downside risk is valued as a European put option on the portfolio return with a strike equal to the disappointment threshold. If the portfolio return at the end of the investment period is below the disappointment threshold, the option matures in the money, reducing the utility of the
investor. The total cost of downside risk is the expected payoff of this put option, \( E[p_W] \), times the degree of disappointment aversion, \( \ell \). Thus, \( \ell \) may be interpretable as the marginal cost of downside risk, as a one-basis-point increase in \( E[p_W] \) translates into an \( \ell \)-basis-point decrease in the investor’s certainty equivalent. When \( \gamma \neq 1 \), the intuition remains the same. The first component in (7) is the log certainty equivalent of the EU investor. The second part is the downside risk penalty, which is non-positive by definition and a decreasing function of the put option payoff.

### 2.2 Model of asset returns

We consider a simple extension to the multivariate normal distribution in order to capture the asymmetry in asset returns. Specifically, we assume that log returns on \( N \) risky assets are described by the model

\[
    r_t = \mu - \sigma \circ \delta + (\sigma \circ \delta) \varepsilon_{0,t} + \left(\sigma \circ \sqrt{\iota - \delta \circ \delta}\right) \circ \varepsilon_t ,
\]

where \( \mu, \sigma, \) and \( \delta \) are \( N \)-dimensional vectors, \( \iota \) is a vector of ones, and \( \circ \) denotes the Schur product (element-wise product) of vectors. The scalar \( \varepsilon_{0,t} \) is a common shock across all assets that follows an exponential distribution with a rate parameter equal to one.\(^4\) The \( N \)-dimensional vector \( \varepsilon_t \) represents asset specific shocks and has a multivariate normal distribution, independent of \( \varepsilon_{0,t} \), with standard normal marginal densities and correlation matrix \( \Psi \). Parameters \( \mu, \sigma, \Psi, \) and \( \delta \) together describe the return generating model. If \( \delta = 0 \), then \( r_t \) follows a multivariate normal distribution with mean \( \mu \), standard deviation

\(^4\) That is, \( \varepsilon_{0,t} \sim \exp(1) \). In a previous version of the paper we considered an alternative model for asset returns, known as the extended skew-normal distribution, where the common shock has a truncated normal distribution. The normal-exponential model in (9) has several advantages. First, the formulas for the return moments are simpler. Second, the extended skew-normal model needs one additional parameter. Third, we are able to derive the exact distribution of multi-period returns for the normal-exponential model, while we have to use approximated distributions if we work with the skew-normal model. Moreover, Adcock and Shutes (2012) show that the normal-exponential model is a certain limiting case of the extended skew-normal distribution, and the two models lead to very similar results in empirical applications.
vector $\sigma$, and correlation matrix $\Psi$. Hence, our setup conveniently nests the case when asset returns are jointly log normal. In our extended model, $N$ additional parameters in $\delta$ are needed compared with the multivariate normal distribution; these additional parameters describe the asymmetry in returns. Note that both Barberis and Huang (2008) and Mitton and Vorkink (2007) consider a case where out of the $N$ risky assets only one asset is skewed, and the others have symmetric returns. The model in (9) nests this scenario by setting $\delta_1 = \ldots = \delta_{N-1} = 0$ and letting only $\delta_N$ to be different from zero.

The log return on asset $i$ may be written as

$$r_{i,t} = \mu_i - \sigma_i \delta_i + (\sigma_i \delta_i) \varepsilon_{0,t} + \left(\sigma_i \sqrt{1 - \delta^2_i}\right) \varepsilon_{i,t} . \quad (10)$$

Parameter $\delta_i$, belonging to the interval $(-1, 1)$, determines the sensitivity of the asset return to the exponentially distributed common shock $\varepsilon_{0,t}$. The exponential distribution is suitable for characterizing the occurrence of extreme events, such as large and infrequent losses. For example, the waiting time until the next event in a Poisson-process has an exponential distribution. The Poisson-process is often used to characterize the occurrence of jumps in continuous-time models (see, e.g., Merton, 1976; Bates, 1996; and Broadie et al., 2007). Assets with large negative sensitivities to $\varepsilon_{0,t}$ are subject to large but infrequent negative returns, while assets with large positive sensitivities are subject to large but infrequent positive returns. Model (9) assumes that the occurrence of such extreme movements is simultaneous across assets, so it may be interpretable as a systemic event. In this sense, our discrete-time return dynamics share the properties of the continuous-time dynamics considered by Das and Uppal (2004).

It is straightforward to show that the mean, variance, skewness, and excess kurtosis of $r_{i,t}$ are given by

$$E(r_{i,t}) = \mu_i , \quad Var(r_{i,t}) = \sigma_i^2 , \quad Skew(r_{i,t}) = 2\delta_i^3 , \quad Xkurt(r_{i,t}) = 6\delta_i^4 . \quad (11)$$
The correlation and coskewness of the returns of asset $i$ and asset $j$ are

$$
\text{Corr}(r_{i,t}, r_{j,t}) = \psi_{ij} \sqrt{1 - \delta_i^2} \sqrt{1 - \delta_j^2} + \delta_i \delta_j ,
$$

$$
\text{Coskew}(r_{i,t}, r_{j,t}) \equiv \frac{E[(r_{i,t} - E(r_{i,t}))^2 (r_{j,t} - E(r_{j,t}))]}{\text{Var}(r_{i,t}) \sqrt{\text{Var}(r_{j,t})}} = 2\delta_i^2 \delta_j .
$$

The formulas in (11) and (12) show how the vector $\delta$ characterizes the asymmetry in particular and the non-normality of returns more generally. The parameters of the distribution can be estimated by the generalized method of moments (GMM) using the moments given in (11) and (12). For the main part of the paper, we consider the case when the investment horizon is one period. That is, risky asset returns over the investment horizon are described by (9). In Section 3.2.3 we also consider the effects of increasing investment horizon.

The asymmetry in asset returns is attributed to a common source of risk in the normal-exponential model (9). Boyer et al. (2010) argue that idiosyncratic skewness is also important in explaining cross-sectional differences in asset returns. In the Online Appendix we discuss a simple extension to the normal-exponential model that accounts for the assets’ idiosyncratic skewness. We also provide some evidence that the main conclusions regarding optimal portfolios do not change much when the extended model is considered.

### 2.3 The optimal portfolio

The second-order Taylor approximation à la Campbell and Viceira (2002) of the portfolio log return is

$$
r_W \approx r_f + w^\top (r_t - r_{ft} + \frac{1}{2} \sigma^2) - \frac{1}{2} w^\top \Sigma w ,
$$

where $r_f$ is the risk-free rate, $w$ is the vector of portfolio weights for risky assets, $\iota$ is a vector of ones, and $\sigma^2$ is the diagonal of the variance-covariance matrix $\Sigma$.\footnote{In the Online Appendix, we find that the approximation works well in our calibration exercise. One can also consider a third-order approximation instead of (13). However, the order of approximation affects only the mean of the portfolio log return ($\mu_W$ from equation (15)), but not the higher moments. We find in the Online Appendix that (13) approximates the true mean of the portfolio log return well and the third-order approximation works well for our calibration exercise.}
are characterized by the return generating model (9), then using the above approximation, the portfolio log return is also characterized by the normal-exponential model

\[ r_W = \mu_W - \sigma_W \delta_W + (\sigma_W \delta_W) \varepsilon_{0,t} + \left(\sigma_W \sqrt{1 - \delta_W^2}\right) \varepsilon_{W,t}, \]

with

\[ \mu_W = r_f + w^\top \left(\mu - r_f \mu + \frac{1}{2} \sigma^2\right) - \frac{1}{2} w^\top \Sigma w, \]

\[ \sigma_W^2 = w^\top \Sigma w, \]

\[ \delta_W = \frac{w^\top (\sigma \circ \delta)}{\sigma_W}, \]

and where \( \varepsilon_{W,t} \) is a standard normal shock independent of \( \varepsilon_{0,t} \). Given our setup, the following proposition describes the optimal portfolio.

**Proposition 2.1** The investor’s optimal asset allocation may be written as

\[ w = \frac{1}{\tilde{\gamma}} \left( w^{\text{MV}} + \tilde{\chi} w^{\text{AV}} \right) \]

where

\[ w^{\text{MV}} \equiv \Sigma^{-1} \left(\mu - r_f \mu + \frac{1}{2} \sigma^2\right) \quad \text{and} \quad w^{\text{AV}} \equiv \Sigma^{-1} (\sigma \circ \delta). \]

Analytical expressions for the coefficients \( \tilde{\gamma} \) and \( \tilde{\chi} \) are given in Appendix B.

**Proof.** See Appendix B.

Our setup leads to a three-fund separation strategy similar to that of Simaan (1993). The investor allocates her wealth to two risky funds and invests the remainder of her wealth in the risk-free asset. We label the first risky fund, \( w^{\text{MV}} \), with “mean-variance” because it is the solution to the mean-variance optimal portfolio problem. Note that the same fund appears in Campbell and Viceira (2002), in the solution of the lognormal model with power utility. The alternative does not improve the approximation considerably.
We label the second risky fund, \( w^{AV} \), with “asymmetry-variance” because its composition depends on the asymmetry vector \( \delta \) and the variance-covariance matrix of the risky asset returns. It is the solution to an asymmetry-variance optimal portfolio problem similar to the mean-variance one.

The weights that the investor assigns to the risky funds are determined by \( \tilde{\gamma} \) and \( \tilde{\chi} \). These coefficients depend not only on the preference parameters (i.e., \( \gamma \), \( \ell \), and \( \kappa \)) but also on the optimal asset allocation, \( w \), itself and the certainty equivalent, \( \eta \). That is, the coefficients \( \tilde{\gamma} \) and \( \tilde{\chi} \) and the certainty equivalent \( \eta \) are all endogenous to the model. To solve for these values and for the optimal allocation, \( w \), equations (16) and (7) must be solved simultaneously.

Given the endogenous values of \( \tilde{\gamma} \) and \( \tilde{\chi} \), the optimal allocation (16) can also be achieved by solving the following mean-variance-asymmetry investment problem:

\[
\max_w \quad \mu_W - r_f - \frac{\tilde{\gamma} - 1}{2} \sigma^2_W + \tilde{\chi} \sigma_W \delta_W,
\]

where \( \mu_W \) and \( \sigma^2_W \) are the mean and variance of the portfolio log return given in equation (15), while \( \delta_W \) describes its asymmetry. Therefore, we can interpret the coefficient \( \tilde{\gamma} \) as the effective risk aversion and the coefficient \( \tilde{\chi} \) as the implicit asymmetry aversion of the investor. The finding that effective risk aversion is endogenous under disappointment-aversion preferences is consistent with the discussions presented by Routledge and Zin (2010) and Bonomo et al. (2011) in an intertemporal consumption-based general equilibrium setting. However, unlike these authors, we explicitly derive the formula of effective risk aversion in our partial equilibrium setting. This provides a novel way to quantify the effect of disappointment aversion on the optimal portfolio choice. The mean-variance-asymmetry problem (18) is similar to that in Mitton and Vorkink (2007) and Harvey et al. (2010), among others, but differs in several ways: the asymmetry measure is not the third central moment of returns; the coefficient \( \tilde{\chi} \), governing preference for asymmetry, is not positive a priori; and as already
shown, our solution is analytical. We show in Appendix C that the three funds in (16) span the mean-variance-asymmetry efficient frontier, which contains portfolios that minimize portfolio variance $\sigma_W^2$, for a given level of mean $\mu_W$ and asymmetry $\sigma_W \delta_W$. Therefore, the optimal portfolios given by (16) are on the efficient frontier.

The lack of asymmetry in asset returns ($\delta = 0$) implies both $w^{AV} = 0$ and $\tilde{\chi} = 0$. Hence, the optimal portfolio rule simplifies to

$$w = \frac{1}{\tilde{\gamma}} w^{MV}.$$  \hspace{1cm} (19)

When returns are symmetric, investors allocate their wealth between the mean-variance fund and the risk-free asset. Consequently, when observing a particular asset allocation, we cannot determine whether it was chosen by a disappointment-averse or a disappointment-neutral investor. In other words, different combinations of the preference parameter values $\gamma$, $\ell$, and $\kappa$ lead to the same $\tilde{\gamma}$. Therefore, the concept of effective risk aversion provides a convenient way to compare the effects of different preferences in the presence of return asymmetries. Comparing the optimal choices of different investors (e.g., power utility versus disappointment-averse investors) who have the same effective risk aversion isolates the effect of return asymmetries, as these investors would choose the same portfolios if returns were symmetric.

If the investor has power utility, the effective risk aversion is simply the risk aversion for the power utility ($\ell = 0$ implies $\tilde{\gamma} = \gamma$). Disappointment aversion ($\ell > 0$), on the other hand, implies $\tilde{\gamma} > \gamma$; that is, a disappointment-averse investor reduces investment in risky assets, investing a larger fraction of wealth in cash.
3 Empirical application

3.1 Data and parameter estimation

In this section we investigate how investors who differ in their degree of risk aversion and
disappointment aversion allocate their wealth among three assets: cash, bonds, and stocks.
We estimate return parameters using monthly data for the USA from July 1952 to December
2012, which we obtained from the Center for Research in Security Prices (CRSP). The risk-
free rate is the average of the log return on the 30-day Treasury bill from the CRSP Fama
Risk-Free Rates file, referred to simply as “cash.” The bond return is the return on the
10-year government bond index from the US Treasury and Inflation Series file in CRSP. The
stock return is the value-weighted return on the NYSE, NASDAQ, and AMEX. The excess
log bond return is the difference between the log return on bonds and the risk-free rate.
Similarly, the excess log stock return is the difference between the log return on stocks and
the risk-free rate.

Table 1 presents estimation results for the return distribution of the two risky assets as
in (9). The parameters are estimated by minimizing the distance between model-implied
moments and their sample counterparts, using the generalized method of moments (GMM)
with an identity-weighting matrix. The table reports three different GMM estimation results.
GMM I is exactly identified and fits the two means, the two volatilities, the correlation, and
the two skewness values, where subscript “$B$” is used for bonds and subscript “$S$” for stocks.
GMM II is overidentified, fitting the two coskewness values in addition to the seven moments
considered in GMM I. GMM III fits the same moments as does GMM I but replaces the
skewness of the bonds with the coskewness of the stocks relative to the bonds. The top
panel of Table 1 presents sample and fitted moments, while the bottom panel presents model
parameter estimates.

The sample moments in Table 1 indicate that the monthly excess log bond return has a
mean of 0.13%, a volatility of 2.12%, and a positive skewness of 0.20, while the monthly excess log stock return has a mean of 0.46%, a volatility of 4.26%, and a negative skewness of -0.64. The correlation between the two risky assets is 0.10. All sample moment estimates are significant at conventional significance levels, except for the skewness of the bonds and the coskewness of the bonds relative to the stocks. GMM III is our preferred parameter configuration as it matches the third-order moments of the joint asset distribution that differ significantly from zero. Note that the implied coskewness of the bonds relative to the stocks exactly matches its data counterpart, but that the implied skewness of the bonds is lower than the sample estimate.

The proposed model of asset returns seems to capture all key asset return moments, providing a simple characterization of the return distribution. In contrast, a multivariate normal model of asset returns would assume that the stock skewness and the coskewness of the stocks relative to the bonds equal zero while the data indicate that these higher moments differ significantly from zero. To further illustrate the ability of the model (9) to match key features of asset returns, we use the parameter estimates in Table 1 to compute, via simulations, two additional statistics. The first statistic is the correlation between the bonds and the stocks conditional on the stocks falling below a given quantile of their distribution. The second statistic is the stocks’ expected shortfall at a given quantile of the stock distribution. These two statistics are plotted in Figure 1, together with their data counterpart and their analogue computed in the normal distribution. The figure confirms that our return generating model (9) fits these features of the data far better than does the multivariate normal model. Moreover, the fit of the GMM III model is closest to the data, corroborating our choice for the calibration assessment.
3.2 Results

Given the estimated return distribution, the mean-variance fund, \( w^{MV} \), and the asymmetry-variance fund, \( w^{AV} \), can be calculated according to (17). Let us normalize each fund by the absolute value of the sum of its weights. The compositions of the normalized funds are

\[
\bar{w}^{MV} = \frac{w^{MV}}{\|\mathbf{l}^\top w^{MV}\|} \quad \bar{w}^{AV} = \frac{w^{AV}}{\|\mathbf{l}^\top w^{AV}\|}.
\]

(20)

The first entry corresponds to the bonds and the second entry to the stocks. The mean-variance fund assigns a positive weight to both risky assets as they have positive expected excess returns. As the stocks are negatively skewed, the asymmetry-variance fund assigns a negative weight to them, while the bond weight is positive.\(^6\) The optimal portfolio rule from (16) can be rewritten as

\[
w = \alpha^{MV} \bar{w}^{MV} + \alpha^{AV} \bar{w}^{AV},
\]

(21)

where

\[
\alpha^{MV} \equiv \frac{1}{\gamma} \|\mathbf{l}^\top w^{MV}\| \quad \text{and} \quad \alpha^{AV} \equiv \frac{\tilde{\chi}}{\gamma} \|\mathbf{l}^\top w^{AV}\|
\]

(22)

are the weights assigned to the normalized mean-variance and asymmetry-variance funds, respectively. The optimal investment in cash is \( 1 - \alpha^{MV} + \alpha^{AV} \). Note that \( \alpha^{MV} \) is a scalar multiple of the investor’s effective risk tolerance (i.e., the inverse of effective risk aversion), while \( \alpha^{AV}/\alpha^{MV} \) is a scalar multiple of the investor’s implicit asymmetry aversion.

Figure 2 summarizes how investors with different preferences choose their optimal portfolios in our calibrated example. The weight assigned to the mean-variance fund, \( \alpha^{MV} \), is on the horizontal axis and the relative weight of the asymmetry-variance fund, \( \alpha^{AV}/\alpha^{MV} \), is on the vertical axis. All curves start at the same point corresponding to the investor with \( \gamma = 2 \) and \( \ell = 0 \). Note that the optimal allocation of this investor is 142.8% in the

\(^6\)Note that the sum of weights in \( \bar{w}^{AV} \) is –100%. Increasing the weight of \( \bar{w}^{AV} \) in the portfolio corresponds to taking a short position in stocks and a long position in cash and bonds.
stock and 136.1% in the bond (and borrowing cash). The solid line corresponds to the EU investor and shows the effect of increasing $\gamma$ from 2 to 30. The other curves correspond to disappointment-averse investors with different $\kappa$ values and show the effect of increasing $\ell$ from 0 to 3, while keeping $\gamma$ fixed at 2. Increasing either $\gamma$ or $\ell$ leads to higher effective risk aversion, $\tilde{\gamma}$, and consequently to a smaller investment in the mean-variance fund, $\alpha_{MV}$. Therefore, increasing $\gamma$ for the EU investor or increasing $\ell$ for the disappointment-averse investor corresponds to moving to the left along the horizontal axis.

Figure 2 provides an overall picture about the endogenous asymmetry preference of different investors (recall that $\alpha_{AV}/\alpha_{MV}$ is a scalar multiple of $\tilde{\chi}$). Let us start with the EU investor. First, the implicit asymmetry aversion is positive for all values of $\gamma$, i.e., the EU investor dislikes negative asymmetry in returns. Second, the relative weight in the asymmetry-variance fund grows as risk aversion increases (or equivalently, the implicit asymmetry aversion increases). However, the magnitude of the increase is modest: the relative weight of the asymmetry-variance fund in the optimal portfolio is 0.18% at $\gamma = 2$, increasing to only 0.26% at $\gamma = 30$. This emphasizes that EU investors with power utility pay relatively little attention to asymmetries in asset returns.

Turning to disappointment-averse investors, let us first consider the special case of the DA investor ($\kappa = 1$). This is the only case from Figure 2 where $\alpha_{MV}$ reaches zero: it becomes zero at $\ell^* = 0.43$ and remains there for all values of $\ell > \ell^*$. That is, a DA investor with high enough disappointment aversion does not hold risky securities at all. This was first discussed by Ang et al. (2005). However, this result does not hold for disappointment-averse investors for whom $\kappa \neq 1$. A GDA investor will always find it optimal to hold some risky assets, regardless of the degree of her disappointment aversion. The difference between DA and GDA investors regarding participation in risky asset markets is discussed in detail by Farago (2014).

Figure 2 reveals that the asymmetry preference of GDA investors is very different when $\kappa < 1$ compared to cases when $\kappa > 1$. For GDA investors with a disappointment threshold
lower than the certainty equivalent ($\kappa < 1$), the relative weight in the asymmetry-variance fund increases considerably as the disappointment aversion $\ell$ increases. That is, the implicit asymmetry aversion increases with $\ell$. The value of $\alpha^{AV}/\alpha^{MV}$ can be as high as 2.6% for GDA investors in Figure 2, which is ten times higher than the value that a highly risk-averse EU investor would choose. This causes a significant shift from stocks to bonds in the optimal portfolio. On the other hand, GDA investors with a disappointment threshold higher than the certainty equivalent ($\kappa > 1$) take a short position in the asymmetry-variance fund, and the relative weight increases as the disappointment aversion $\ell$ grows. That is, the implicit asymmetry aversion of these investors is negative. The value of $\alpha^{AV}/\alpha^{MV}$ can get as low as $-2\%$ causing a shift from bonds to stocks in the optimal portfolio.

To get a deeper understanding of why asymmetry preference differs so much across different disappointment-averse investors, Table 2 presents details of the choice corresponding to different sets of preference parameters. The table presents GDA investors with $\gamma = 2$, $\ell = 2$, and $\kappa$ ranging from 0.96 to 1.04. The choice of the DA investor ($\kappa = 1$) is not presented as she chooses not to participate in risky asset markets when $\ell = 2$. To highlight the effect of GDA preferences, Panel B describes the choice of the EU investor who has the same effective risk aversion as does the corresponding GDA investor in the same column. Consequently, their investment in the mean-variance fund is exactly the same and the difference between their optimal portfolios comes from the weights they assign to the asymmetry-variance fund.

Let us focus first on the cases where $\kappa \leq 1$. The disappointment threshold of these GDA investors is negative and decreases by roughly one percentage point when $\kappa$ decreases by 0.01. That is, disappointment is connected to left tail events. For example, the threshold is $-1.54\%$ for the investor with $\kappa = 0.98$. Since the GDA investor has a strong focus on avoiding disappointment, she dislikes assets that contribute greatly to the portfolio’s left-tail risk. The probability that the stock return falls below $-1.54\%$ in a given month is 25.6%, while the probability that the bond return falls below this threshold is only 16.8%.\(^7\) Therefore, the

\(^7\)The disappointment probability of an asset for a given GDA investor is $\pi_i \equiv Pr[r_{i,T} \leq \ln \kappa + \eta]$, where
GDA investor shifts from the stocks towards the bonds in her risky portfolio. Consequently, the disappointment probability of the optimal portfolio for the GDA investor with $\kappa = 0.98$ is 2.5%, which is much lower than the 4.3% probability that the comparable EU investor’s portfolio return falls below the same threshold is. Looking at the composition of the optimal portfolios, both the GDA and the comparable EU investor put 46.9% of their wealth in the mean-variance fund $\bar{w}^{\text{MV}}$, but they assign different weights to the asymmetry-variance fund. The GDA investor assigns 1.19% of her wealth to $\bar{w}^{\text{AV}}$, while the EU investor assigns only one tenth of this amount, 0.11%. As a result, both of them have a similar amount invested in cash, but the GDA investor’s bond weight is 4.6 percentage points higher and stock weight is 5.7 percentage points lower than those of the comparable EU investor.

When the disappointment threshold of the GDA investor is lower, she focuses only on avoiding relatively large losses. The disappointment threshold of the investor with $\kappa = 0.96$ is -3.51%. The probability that the bond or the stock return falls below this threshold is 2.9% and 14.1%, respectively. The relative difference in disappointment probability gets bigger as we move further out in the left tail, hence the shift towards bonds by the GDA investor becomes more pronounced. She invests 2.25% in $\bar{w}^{\text{AV}}$, an EU investor with the same effective risk aversion invests only 0.18%. Consequently, the GDA investor’s bond weight is 8.5 percentage points higher and stock weight is 10.5 percentage points lower than those of the comparable EU investor.

In general, negative asset skewness is associated with an increased probability of relatively bigger losses. Therefore, a GDA investor with a focus on avoiding these losses will take a large long position (compared to the EU investor) in the asymmetry-variance fund. Hence, the implicit asymmetry aversion these GDA investors ($\kappa < 1$) is positive.

Now let us consider the cases when $\kappa > 1$. The disappointment threshold of the investor $i \in \{S, B\}$ refers to the stock or the bond. Similarly, the disappointment probability of the optimal portfolio is $\pi_W = Pr[r_{W,T} \leq \ln \kappa + \eta]$. Note that the comparable EU investor does not become disappointed, but we can calculate the probability that her portfolio return is below the disappointment threshold ($\ln \kappa + \eta$) of the corresponding GDA investor.
increases with $\kappa$. The GDA investor with $\kappa = 1.02$, for example, is satisfied only if the portfolio return is at least 2.51%. To achieve such high returns, she has to rely on assets making a greater contribution to the portfolio’s upside. The probability that the stock return is higher than 2.51% is 36.4%, much higher than the 17.2% probability that the bond return exceeds this threshold.\(^8\) Therefore, the GDA investor shifts from bonds to stocks in her risky portfolio by taking a short position in the asymmetry-variance fund. The GDA investor with $\kappa = 1.02$ assigns a $-1.13\%$ weight to the asymmetry-variance fund, while the comparable EU investor assigns $0.13\%$. Consequently, the GDA investor’s stock weight is 6.7 percentage points higher and her bond weight is 5.4 percentage points lower than those of the EU investor. This leads to a higher probability of non-disappointment, 9.4% for the GDA versus 6.2% for the EU investor.

By preferring the stocks because they have greater contribution to the portfolio’s upside, GDA investors with $\kappa > 1$ take a short position in the asymmetry-variance fund. This leads to negative implicit asymmetry aversion. However, it is important to emphasize that these investors do not explicitly like negative skewness, but their preference for the stock is represented by negative $\tilde{\chi}$ values. The contrast between GDA investors with $\kappa < 1$ and GDA investors with $\kappa > 1$ may also be connected to the contrast between hedgers and speculators. Consider, for example, the GDA investors with $\kappa = 0.96$ and $\kappa = 1.04$ from Table 2. They have very similar effective risk aversions, but their portfolios differ greatly: the $\kappa = 0.96$ investor has a bond/stock allocation ratio of 1.52, while the bond/stock ratio of the $\kappa = 1.04$ investor is 0.68.

So far, we have looked at how optimal portfolios change with the preference parameter values. Now, let us study the sensitivity of the optimal portfolio weights to the asymmetry in returns. In our benchmark calibration, we estimate the stock skewness to be $-0.64$. However, return asymmetries exhibit significant time variation and there might be periods with much larger skewness. Neuberger (2012), for example, estimates that expected skewness of the

\(^8\)These probabilities can be calculated as one minus the disappointment probability reported in Table 2.
S&P 500 index at the quarterly horizon varies between –1.8 and –1.0 over the period from 1998 to 2010. In Figure 3 we use the same distributional parameters as in the main calibration (corresponding to GMM III in Table 1), but vary the asymmetry parameter of the stock, $\delta_S$, so that stock skewness varies between –1.8 and –0.3. The vertical line corresponds to our benchmark calibration. The figure presents optimal portfolio weights for an EU investor and two GDA investors, one with $\kappa < 1$ and another one with $\kappa > 1$. Optimal portfolio weights do not vary much for the EU investor, providing further evidence that return asymmetries do not have a large effect on the portfolio choice of power utility investors. The GDA investor with a low disappointment threshold further reduces her stock investment and increases her bond investment when the stock becomes more negatively skewed. At a stock skewness of –1.8, her stock and bond weights are 26% and 61%, respectively. To the contrary, the GDA investor with a high disappointment threshold further shifts her risky portfolio from bonds towards stocks even further as the stock skewness becomes more pronounced. At –1.8 skewness, her stock and bond weights are 82% and 17%, respectively.

3.2.1 Costs of ignoring skewness

Following Das and Uppal (2004), we quantify the certainty-equivalent cost of ignoring return asymmetries. An investor who ignores asymmetry in the distribution of asset returns and chooses her optimal portfolio as if log asset returns were normally distributed with the same mean and variance-covariance matrix as the true distribution, chooses allocation $w'$. That suboptimal allocation corresponds to a certainty equivalent, $R'$, under the true distribution. The cost of ignoring skewness can be measured in absolute terms by

$$R - R'$$

---

9The EU investor has $\gamma = 6.3$, corresponding to the first column in Panel B of Table 2. The first GDA investor has preference parameter values $\gamma = 2$, $\ell = 2$, and $\kappa = 0.96$. This is the investor in the first column in Panel A of Table 2. The second GDA investor has preference parameter values $\gamma = 2$, $\ell = 2$, and $\kappa = 1.04$. This is the investor in the last column in Panel A of Table 2.
or in relative terms by
\[ \frac{R' - R_f}{R - R_f}. \]

The above ratio is the excess certainty equivalent of the suboptimal allocation relative to the excess certainty equivalent of the optimal allocation.

Table 3 shows the absolute and relative costs of ignoring skewness for different investors using the annualized values of the certainty equivalents \( R \) and \( R' \). Note that the absolute measure is multiplied by 1000, so that it indicates the cost for an investor with an initial wealth of $1000. For EU investors, the certainty-equivalent cost of ignoring skewness is almost negligible, the annualized cost being less than $0.01 in all the cases. This is in line with the findings of Das and Uppal (2004).\footnote{Das and Uppal (2004) measure the cost of ignoring systemic risk for EU investors in an international portfolio choice setting. They find similar costs for an investor with a relative risk aversion of five and a one-year horizon when the portfolio consists of equity indexes of developed countries.} The relative measures indicate that EU investors achieve more than 99.9% of the overall optimal excess certainty equivalent even if they ignore return skewness.

For GDA investors, the cost of ignoring skewness is more substantial. When \( \kappa = 0.96 \), the cost is $2.68, considerably higher than for the comparable EU investor. In relative terms, this investor achieves only 89.2% of the optimal excess certainty equivalent if she ignores return skewness. As \( \kappa \) increases, the cost of ignoring return asymmetries declines, but it is still much higher for all GDA investors than for the comparable EU investors.

### 3.2.2 The asset-allocation puzzle of Canner et al. (1997)

The two-fund separation strategy arising from standard models implies that all investors should hold risky assets in the same proportion, and should change only their relative weights in the risky portfolio and in cash according to their risk appetite. Consequently, all investors should have the same bond/stock ratio in their portfolios. The asset-allocation puzzle of Canner et al. (1997) is that, in contrast with the above theoretical predictions, financial
advisors recommend different ratios for different investors: a high bond/stock ratio for "conservative" investors and a low ratio for "aggressive" investors. Table 4 is adapted from Canner et al. (1997) and presents the recommendations of four financial advisors, together with the assumed asset returns from the original paper.\textsuperscript{11} For each advisor, the bond/stock ratio ($w_B/w_S$) increases as we move from "aggressive" toward "conservative" portfolios.

Given the distributional assumptions in Panel A of Table 4, we can determine the composition of the mean-variance and asymmetry-variance funds using (17). The resulting normalized funds, $\bar{w}^{\text{MV}}$ and $\bar{w}^{\text{AV}}$ are given in the last two columns of Panel A. Equation (21) shows that each recommended portfolio can be constructed using these normalized funds. Since there are two equations (one for the stock weight, $w_S$, and one for the bond weight, $w_B$), there is a unique pair of fund weights, $\alpha^{\text{MV}}$ and $\alpha^{\text{AV}}$, that yields a given recommended portfolio. We calculate these weights for each recommended portfolio and the last two columns in Panel B show the corresponding weight in the mean-variance fund ($\alpha^{\text{MV}}$) and the relative weight in asymmetry-variance fund ($\alpha^{\text{AV}}/\alpha^{\text{MV}}$). As we move from "aggressive" towards "conservative" portfolios, the weight in the mean-variance fund decreases, which is consistent with increasing effective risk aversion. At the same time, the relative weight in the asymmetry-variance fund increases, leading to an increase in the bond/stock allocation ratio and consistent with increasing implicit asymmetry aversion. Looking at Figure 2, we see that the $\alpha^{\text{AV}}/\alpha^{\text{MV}}$ values corresponding to "moderate" and "conservative" portfolios are much higher than those implied by EU preferences, though they are in line with the choice of GDA investors for whom $\kappa < 1$. That is, asset skewness together with increasing disappointment aversion from "aggressive" to "conservative" investors offers an explanation for the asset allocation puzzle of Canner et al. (1997). Finally, note that "aggressive" portfolios are typically associated with negative $\alpha^{\text{AV}}/\alpha^{\text{MV}}$ values. As we have discussed earlier, the speculative behavior of GDA investors for whom $\kappa > 1$ might lead to optimal portfolios with

\textsuperscript{11}Canner et al. (1997) do not consider asset skewness. We use $s_B = 0.02$ and $s_S = -0.64$ as estimated in our data, but the results are not sensitive to moderate changes in these values.
this characteristic.

### 3.2.3 The effect of time horizon

We next examine the effect of the investment horizon on optimal portfolios assuming that the one-period returns are IID. Consider the $H$-period log return between dates $t$ and $t + H$,

$$
r_{t, t+H} \equiv \sum_{h=1}^{H} r_{t+h} . \tag{23}$$

It can be shown that the $H$-period returns follow\(^{12}\)

$$
r_{t, t+H} = \mu_H - \sqrt{H} (\sigma_H \circ \delta) + (\sigma_H \circ \delta) \, \varepsilon_{0, t, t+H} + \left(\sigma_H \circ \sqrt{t - \delta \circ \delta}\right) \circ \varepsilon_{t, t+H} , \tag{24}$$

where,

$$
\varepsilon_{0, t, t+H} \sim \Gamma \left( H, 1/\sqrt{H} \right) , \quad \varepsilon_{t, t+H} \sim N \left( 0, \Psi \right) , \quad \mu_H = H \mu , \quad \sigma_H = \sqrt{H} \sigma . \tag{25}$$

Note that the parameters $\delta$ and $\Psi$ do not have a $H$ subscript, since their values are independent of the horizon. The moments of the $H$-period return of asset $i$ are given by

$$
\begin{align*}
E \left( r_{i, t, t+H} \right) &= H \mu_i , \\
Var \left( r_{i, t, t+H} \right) &= H \sigma_i^2 , \\
Skew \left( r_{i, t, t+H} \right) &= \frac{2 \delta_i^3}{\sqrt{H}} , \\
Xkurt \left( r_{i, t, t+H} \right) &= \frac{6 \delta_i^4}{H} .
\end{align*} \tag{26}
$$

Both the mean and variance of the assets grows by $H$ as the horizon increases. Asset skewness, on the other hand, is scaled by $1/\sqrt{H}$. That is, skewness diminishes as $H$ increases and the distribution of long-horizon returns is closer to normal than the distribution of short-horizon returns. In fact, for large values of $H$ the distribution of the common shock $\varepsilon_{0, t, t+H}$

\(^{12}\)The key result for deriving the return generating model for the $H$-period returns is that the sum of $H$ IID exponential variables is a random variable that follows a gamma distribution with a shape parameter $H$. Note also that for $H = 1$, the model in (24) is equivalent to the model in (9), since $\Gamma \left( 1, 1 \right) \sim \exp \left( 1 \right)$. 

27
converges to a normal distribution. Hence, the asset log returns become jointly normal.

The optimal asset allocation for an investor with horizon $H$ may be written as

$$w_H = \frac{1}{\tilde{\gamma}_H} \left( w^{MV}_H + \tilde{\chi}_H w^{AV}_H \right).$$

(27)

Note that $w^{MV}$ does not have a $H$ subscript. The mean-variance fund has the same composition, irrespective of the investor’s horizon. Although the second risky fund, $w^{AV}_H$, does have a time subscript, it can be shown that $w^{AV}_H = w^{AV}/\sqrt{H}$. That is, investors with different horizons will use the same asymmetry-variance fund in their portfolios, only the size of their investment will be different. The parameters describing the risk attitude of the investor, $\tilde{\gamma}_H$ and $\tilde{\chi}_H$, depend on the horizon.\(^{13}\)

For a given level of effective risk aversion, $\tilde{\gamma}_H$, the only part of the optimal portfolio rule (21) that changes with the investment horizon is $\alpha^{AV}$, i.e., the weight assigned to $w^{AV}$. To illustrate the effect of horizon in our calibration, for each $H$ we fix $\tilde{\gamma}_H = 5$ (consequently, fix $\alpha^{MV}$) and calculate the corresponding weight in the asymmetry-variance fund for different investors. For an EU investor, $\tilde{\gamma}_H = 5$ implies $\gamma = 5$. For a GDA investor, a given value of $\tilde{\gamma}_H$ can correspond to different sets of parameter values. We fix $\gamma = 2$ and $\ell = 2$, and choose the value of $\kappa$ that leads to $\tilde{\gamma}_H = 5$. Note that there are two $\kappa$ values that lead to $\tilde{\gamma}_H = 5$, one such that $\kappa < 1$ and the other such that $\kappa > 1$. We report results for both cases.

Figure 4A shows how $\alpha^{AV}/\alpha^{MV}$, the relative weight in the normalized asymmetry-variance fund, changes with the horizon. Return asymmetries do not have a large effect on the EU investor’s portfolio as her relative weight barely changes with $H$. For a GDA investor, however, the investment horizon is an important factor determining the optimal portfolio. Over short horizons, GDA investors hold different $\alpha^{AV}/\alpha^{MV}$ ratios than do EU investors due to asymmetries in returns. However, these return asymmetries become less pro-

\(^{13}\)Analytical formulas for the horizon-dependent effective risk aversion $\tilde{\gamma}_H$ and implicit asymmetry aversion $\tilde{\chi}_H$ are given in the Online Appendix.
nounced as the horizon increases. Hence, the $\alpha^{AV}/\alpha^{MV}$ of the GDA investors approaches that of the EU investors. Disappointment aversion (with $\kappa < 1$) together with skewness prompts a shift from bonds to stocks as the investment horizon increases. This is a different mechanism from the often-emphasized effect of mean reversion in prices (see, e.g., Campbell and Viceira, 2002 and 2005) or non-tradable human capital (see, e.g., Jagannathan and Kocherlakota, 1996; and Cocco et al., 2005).

When returns are IID, the effect of skewness quickly disappears as the horizon increases. However, there is evidence that return skewness does not diminish with the investment horizon as quickly as implied by the IID assumption. To illustrate the effect of persistence in skewness, instead of relying on the IID assumption to calculate $H$-period returns, we fit our return generating model (9) to returns aggregated over $H = 1, ..., 12$ months. Figure 4B shows the stock’s skewness over different horizons. The sample estimates are further from zero than the values implied by the IID assumption. In fact, the skewness of the $H$-month return is greater in magnitude than that of the one-month return for all $H > 1$. Figures 4C and 4D show the optimal stock weight and bond/stock allocation ratio, respectively, for the EU investor and the GDA investor with $\kappa < 1$ in the IID case, and for the same GDA investor with the estimated $H$-period returns. Since the return asymmetry in the last case does not vanish as the horizon increases, the optimal choice of the GDA investor does not converge to that of the EU investor for holding periods up to one year. This evidence suggests that return asymmetry has a larger effect on optimal portfolios for longer investment horizons than in an IID calibration.

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14Neuberger (2012) develops an unbiased estimate of the third moment of long-horizon returns from high-frequency returns. He finds that the skewness of US equity index returns does not diminish with the horizon; it actually increases with horizons up to a year and its magnitude is economically important. Ghysels et al. (2014) introduce an asymmetry measure based on conditional quantiles and find that the return asymmetry is more pronounced at the quarterly frequency than at the monthly frequency for the US and many other countries in their sample.
4 Conclusion

We study the joint impact of two types of asymmetries on investor portfolio choice: asymmetries in asset returns and in investor attitudes towards risk. We model asymmetric investor preferences according to generalized disappointment aversion, and asymmetric return distributions using a normal-exponential model.

We find that these two types of asymmetries jointly yield qualitatively different optimal portfolios from those of the standard model in which both these asymmetries are ignored. On the one hand, when asset returns are symmetric, all investors hold the same risky portfolio. Consequently, when observing a particular asset allocation, it is impossible to determine whether it was chosen by a disappointment-averse or a disappointment-neutral investor. On the other hand, standard preferences imply that return asymmetry only marginally affects the composition of optimal portfolios. However, when both asymmetries are taken into account, the composition of the optimal portfolio changes. In our calibrated example, a disappointment-averse investor with a reference point lower than the certainty equivalent of the investment shifts from the negatively skewed stocks towards the bonds in order to avoid the occasional large losses that negatively skew the stock returns. We also demonstrate that the portfolio choice of an investor with longer investment horizon is less affected by skewness.
Appendix

A The log certainty equivalent

Recall that $r_W = \ln R_W$, $\eta = \ln (R)$, and $U(X) = \frac{X^{1-\gamma}}{1-\gamma}$. We can rewrite

$$U(\kappa R) - U(R_W) = U(\kappa R) \left(1 - \frac{U(R_W)}{U(\kappa R)}\right) = U(\kappa R) \left(1 - \left(\frac{R_W}{\kappa R}\right)^{1-\gamma}\right)$$

$$= \kappa^{1-\gamma} U(R) (1 - \exp ((\gamma - 1)(\ln \kappa + \eta - r_W))).$$

(A.1)

Noting that $\forall a, X \in \mathbb{R}$

$$(1 - \exp (aX)) I(X > 0) = 1 - \exp (aX I(X > 0)) = 1 - \exp (a \max (X, 0)),$$  

(A.2)

equation (A.1) implies

$$E[(U(\kappa R) - U(R_W)) I(R_W < \kappa R)] = \kappa^{1-\gamma} U(R)(1 - E[\exp ((\gamma - 1)p_W)])$$

(A.3)

where $p_W \equiv \max (\ln \kappa + \eta - r_W, 0)$.

Substituting (A.3) into (6) and solving for $U(R)$, we arrive at

$$U(R) = \frac{E[U(R_W)]}{\theta + \ell \kappa^{1-\gamma} (1 - E[\exp ((\gamma - 1)p_W)])}.$$  

(A.4)

$$\ln R^{1-\gamma} = \ln E \left[R_W^{1-\gamma}\right] - \ln \left(\theta + \ell \kappa^{1-\gamma} (1 - E[\exp ((\gamma - 1)p_W)])\right).$$

This finally leads to the first case in equation (7). The second case in (7) derives directly from the first case by taking the limit and applying l’Hôpital’s rule.
B Proof of Proposition 2.1

Equation (7) defines an implicit function

\[ G(w, \eta) \equiv -\eta + \frac{1}{1 - \gamma} \ln E[\exp ((1 - \gamma) r_W)] - \frac{1}{1 - \gamma} \ln (\theta + \ell \kappa^{1-\gamma} (1 - E[\exp ((\gamma - 1) p_W)])) = 0. \] (A.5)

Implicit differentiation of (A.5) implies that

\[ \frac{\partial \eta}{\partial w} = -\frac{G'_1(w, \eta)}{G'_2(w, \eta)}, \] (A.6)

where \( G'_1 \) is the partial derivative of \( G \) with respect to its first argument and \( G'_2 \) is the partial derivative of \( G \) with respect to its second argument. If an optimal allocation policy exists, it satisfies the necessary condition \( \frac{\partial \eta}{\partial w} = 0 \), implying

\[ G'_1(w, \eta) = 0. \] (A.7)

From (A.5),

\[ G'_1(w, \eta) = \frac{E[\exp ((1 - \gamma) r_W) (\partial r_W / \partial w)]}{E[\exp ((1 - \gamma) r_W)]} - \frac{\ell \kappa^{1-\gamma} E[\exp ((\gamma - 1) p_W) (\partial p_W / \partial w)]}{\theta + \ell \kappa^{1-\gamma} (1 - E[\exp ((\gamma - 1) p_W)])}. \] (A.8)

Equation (13) implies

\[ \frac{\partial r_W}{\partial w} = \left( r_t - r_{ft} + \frac{1}{2} \sigma^2 \right) - \Sigma w \quad \text{and} \quad \frac{\partial p_W}{\partial w} = -\frac{\partial r_W}{\partial w} I(r_W < \ln \kappa + \eta), \] (A.9)

which substituting into (A.8) yields

\[ G'_1(w, \eta) = \frac{E[\exp ((1 - \gamma) r_W) r_t]}{E[\exp ((1 - \gamma) r_W)]} + \frac{\nu}{1 - \nu} \frac{E[\exp ((1 - \gamma) r_W) r_t I(r_W < \ln \kappa + \eta)]}{E[\exp ((1 - \gamma) r_W)]} + \frac{1}{1 - \nu} \left( -r_{ft} + \frac{1}{2} \sigma^2 - \Sigma w \right). \] (A.10)
where

\[ \nu \equiv \ell \kappa^{1-\gamma} \exp \left( (\gamma - 1) (\ln \kappa + \eta) \right) \frac{E \left[ \exp \left( (1 - \gamma) r_W \right) I (r_W < \ln \kappa + \eta) \right]}{\theta + \ell \kappa^{1-\gamma} E \left[ I(r_W < \ln \kappa + \eta) \right]} \].

(A.11)

Define

\[ M(u, v; x) \equiv E \left[ \exp \left( u r_W + v^\top r_t \right) I(r_W < x) \right]. \]

(A.12)

Then, (A.10) can be rewritten as

\[ G'_1(w, \eta) = \frac{M'_2(1 - \gamma, 0; \infty)}{M(1 - \gamma, 0; \infty)} + \frac{\nu}{1 - \nu} \frac{M'_2(1 - \gamma, 0; \ln \kappa + \eta)}{M(1 - \gamma, 0; \ln \kappa + \eta)} + \frac{1}{1 - \nu} \left( -r_{ft} + \frac{1}{2} \sigma^2 - \Sigma \right), \]

(A.13)

while the log certainty equivalent and \( \nu \) can be rewritten as

\[ \eta = \frac{1}{1 - \gamma} \ln M((1 - \gamma), 0; \infty) - \frac{1}{1 - \gamma} \ln \left( \theta + \ell \kappa^{1-\gamma} M(0, 0; \ln \kappa + \eta) - \ell \kappa^{1-\gamma} e^{-(1-\gamma)(\ln \kappa + \eta)} M(1 - \gamma, 0; \ln \kappa + \eta) \right), \]

(A.14)

and

\[ \nu = \frac{\ell \kappa^{1-\gamma} e^{-(1-\gamma)(\ln \kappa + \eta)} M(1 - \gamma, 0; \ln \kappa + \eta)}{\theta + \ell \kappa^{1-\gamma} M(0, 0; \ln \kappa + \eta)}. \]

(A.15)

Finding an analytical formula for \( M(u, v; x) \) and \( M'_2(u, v; x) \) allows us to calculate all the quantities of interest from (A.13), (A.14), and (A.15). The Online Appendix contains the derivation of the formulas for these quantities in a more general case, when the investment horizon is \( H \) periods, and the \( H \)-period log return on risky assets between dates \( t \) and \( t + H \) is defined as

\[ r_{t,t+H} \equiv \sum_{h=1}^{H} r_{t+h}. \]

(A.16)

In Proposition 2.1 we consider the case when \( H = 1 \) (and \( r_{t,t+1} \) is simply referred to as \( r_t \)).

Using the results in the Online Appendix for \( H = 1 \),

\[ M(1 - \gamma, 0; \ln \kappa + \eta) = \exp \left( (1 - \gamma) (\mu_W - \sigma_W \delta_W) + \frac{(1 - \gamma)^2 \sigma_W^2 (1 - \delta_W^2)}{2} \right) \frac{\Phi(c_0)}{c_2}. \]

(A.17)
where \( \Phi (\cdot) \) is the cumulative distribution function of the standard normal distribution, and

\[
C \equiv \begin{cases} 
\exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \Phi \left( -\frac{c_2 + c_0c_1}{c_1} \right) & \text{if } c_1 > 0 \\
-\exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \Phi \left( \frac{c_2 + c_0c_1}{c_1} \right) & \text{if } c_1 < 0 
\end{cases}
\]

(A.18)

and

\[
c_0 \equiv \frac{\ln \kappa + \eta - \mu_W + \sigma_W \delta_W - (1 - \gamma) \sigma_W^2 (1 - \delta_W^2)}{\sigma_W \sqrt{1 - \delta_W^2}}
\]

\[
c_1 \equiv \frac{-\delta_W}{\sqrt{1 - \delta_W^2}}
\]

(A.19)

\[
c_2 \equiv 1 - (1 - \gamma) \sigma_W \delta_W .
\]

Note also that

\[
M (1 - \gamma, 0; \infty) = \exp \left( (1 - \gamma) (\mu_W - \sigma_W \delta_W) + \frac{(1 - \gamma)^2 \sigma_W^2 (1 - \delta_W^2)}{2} \right) \frac{1}{c_2} .
\]

(A.20)

It is also shown in the Online Appendix that

\[
\frac{M' (1 - \gamma, 0; \ln \kappa + \eta)}{M (1 - \gamma, 0; \ln \kappa + \eta)} = \mu + \left( 1 - \gamma + \frac{\xi_{\Sigma,0}^B}{\Phi (c_0) + C} \right) \Sigma + \left( (1 - \gamma)^2 \frac{\sigma_W^2 \delta_W^2}{c_2} + \frac{\xi_{\alpha,0}^B}{\Phi (c_0) + C} \right) (\sigma \circ \delta) ,
\]

(A.21)

with

\[
\xi_{\alpha,0}^B = \exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \Phi \left( -\frac{c_2 + c_0c_1}{c_1} \right) \left( -c_2 - \frac{c_2 + c_0c_1}{c_1^2} \right)
\]

\[
+ \exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \phi \left( -\frac{c_2 + c_0c_1}{c_1} \right) \left( \frac{1}{c_1} + c_1 \right) - c_1 \phi (c_0)
\]

\[
\xi_{\Sigma,0}^B = \exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \Phi \left( -\frac{c_2 + c_0c_1}{c_1} \right) \frac{c_2}{\sigma_W \delta_W}
\]

\[
+ \exp \left( \frac{c_2^2 + 2c_0c_1c_2}{2c_1^2} \right) \phi \left( -\frac{c_2 + c_0c_1}{c_1} \right) \frac{1}{\sigma_W \sqrt{1 - \delta_W^2}} - \frac{\phi (c_0)}{\sigma_W \sqrt{1 - \delta_W^2}}
\]

(A.22)

where \( \phi (\cdot) \) is the probability density function of the standard normal distribution. Also,

\[
\frac{M' (1 - \gamma, 0; \infty)}{M (1 - \gamma, 0; \infty)} = \mu + (1 - \gamma) \Sigma + \frac{(1 - \gamma)^2 \sigma_W^2 \delta_W^2}{c_2} (\sigma \circ \delta)
\]

(A.23)
Substituting (A.21) and (A.23) in (A.13), setting it to zero, and solving for \( w \), we can arrive at the optimal portfolio rule

\[
w = \frac{1}{\tilde{\gamma}} \left( \Sigma^{-1} \left( \mu - r_f + \frac{1}{2} \sigma^2 \right) + \tilde{\chi} \Sigma^{-1} (\sigma \circ \delta) \right), \tag{A.24}
\]

with

\[
\begin{align*}
\tilde{\gamma} &= \gamma - \nu \frac{\xi^B_{\Sigma,0}}{\Phi (c_0) + C} \\
\tilde{\chi} &= \frac{(1 - \gamma)^2 \sigma_W^2 \delta_W^2}{1 - (1 - \gamma) \sigma_W \delta_W} + \nu \frac{\xi^B_{\sigma,0}}{\Phi (c_0) + C},
\end{align*}
\tag{A.25}
\]

which corresponds to (16) in the paper. Note that \( \ell = 0 \) implies \( \nu = 0 \), which can easily be seen from (A.15).

## C The efficient frontier

The mean-variance-asymmetry efficient frontier contains portfolios that minimize portfolio variance \( \sigma_W^2 \), for a given level of mean \( \mu_W \) and asymmetry \( \sigma_W \delta_W \). The formulas for these moments are given in (15). The Lagrangian of the problem is

\[
L = \frac{1}{2} w^\top \Sigma w + \lambda_1 \left[ \mu_W - r_f - w^\top \left( \mu - r_f + \frac{1}{2} \sigma^2 \right) + \frac{1}{2} w^\top \Sigma w \right] + \lambda_2 \left[ \sigma_W \delta_W - w^\top (\sigma \circ \delta) \right]. \tag{A.26}
\]

The FOC with respect to \( w \) leads to

\[
w = \frac{\lambda_1}{1 + \lambda_1} \Sigma^{-1} \left( \mu - r_f + \frac{1}{2} \sigma^2 \right) + \frac{\lambda_2}{1 + \lambda_1} \Sigma^{-1} (\sigma \circ \delta). \tag{A.27}
\]

That is, the mean-variance-asymmetry efficient frontier is spanned by the risk-free asset, the mean-variance fund, and the asymmetry-variance fund. Pre-multiplying (A.27) by
\((\mu - r_f t + \frac{1}{2} \sigma^2)^T, (\sigma \circ \delta)^T, \text{ and } w^T \Sigma, \text{ respectively, we get}\)

\[
\begin{align*}
\mu_W - r_f + \frac{1}{2} \sigma^2_W &= \frac{\lambda_1}{1 + \lambda_1} \left( V + \frac{\lambda_2}{\lambda_1} S \right) \\
\sigma_W \delta_W &= \frac{\lambda_1}{1 + \lambda_1} \left( S + \frac{\lambda_2}{\lambda_1} A \right) \\
\sigma^2_W &= \frac{\lambda_1}{1 + \lambda_1} \left( \left( \mu_W - r_f + \frac{1}{2} \sigma^2_W \right) + \frac{\lambda_2}{\lambda_1} \sigma_W \delta_W \right),
\end{align*}
\]  

(A.28)

where

\[
\begin{align*}
V &\equiv \left( \mu - r_f t + \frac{1}{2} \sigma^2 \right)^T \Sigma^{-1} \left( \mu - r_f t + \frac{1}{2} \sigma^2 \right) \\
S &\equiv \left( \mu - r_f t + \frac{1}{2} \sigma^2 \right)^T \Sigma^{-1} (\sigma \circ \delta) \\
A &\equiv (\sigma \circ \delta)^T \Sigma^{-1} (\sigma \circ \delta) .
\end{align*}
\]  

(A.29)

Combining the equations in (A.28), after some algebraic manipulation we arrive at

\[
\sigma^2_W = \frac{A (\mu_W - r_f + \frac{1}{2} \sigma^2_W)^2 - 2S (\mu_W - r_f + \frac{1}{2} \sigma^2_W) (\sigma_W \delta_W) + V (\sigma_W \delta_W)^2}{AV - S^2},
\]  

(A.30)

which implicitly defines the efficient frontier. Interestingly, by denoting the Sharpe ratio of a given portfolio as

\[
\lambda_W \equiv \frac{\mu_W - r_f + \frac{1}{2} \sigma^2_W}{\sigma_W},
\]  

(A.31)

equation (A.30) can be rewritten as

\[
1 = \frac{1}{A} \left[ \frac{(A \lambda_W - S \delta_W)^2}{AV - S^2} + \delta^2_W \right].
\]  

(A.32)

That is, the efficient frontier can be equivalently represented in the asymmetry–Sharpe ratio space.
References


Table 1: Parameter estimates
The table presents parameter and moment estimates for the calibration of the model described in (9). The data used for the calibration are monthly log returns on three assets: 30-day Treasury Bills ($f$), the 10-year government bond index ($B$), and the value-weighted index of the CRSP stocks ($S$). The period used is from July 1952 to December 2012. The first two columns present sample moment estimates together with their bootstrapped 95% confidence intervals. The rest of the table shows the results of three different GMM estimations. GMM I is exactly identified and fits the two means ($\mu$), two volatilities ($\sigma$), correlation ($corr$), and two skewnesses ($skew$). GMM II is overidentified, fitting the two coskewnesses ($coskew$) in addition to the seven moments considered in GMM I. Finally, GMM III fits the same moments as does GMM I, but the skewness of bonds is replaced with the coskewness of stocks relative to bonds. None of the estimations fits the excess kurtosis values ($xkurt$). The top panel of the table shows sample moments and fitted moments. Values with superscript $i$ are not estimated, but are implied by the fitted distribution. The bottom panel shows additional parameter estimates that are needed to fully describe the model in (9).

<table>
<thead>
<tr>
<th></th>
<th>Sample</th>
<th>GMM I</th>
<th>GMM II</th>
<th>GMM III</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Est</td>
<td>95% c.i.</td>
<td>Est</td>
<td>s.e.</td>
</tr>
<tr>
<td>$r_f$ (%)</td>
<td>0.38</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_B - r_f$ (%)</td>
<td>0.13 (0.01, 0.26)</td>
<td>0.13 (0.08)</td>
<td>0.13 (0.08)</td>
<td>0.13 (0.08)</td>
</tr>
<tr>
<td>$\mu_S - r_f$ (%)</td>
<td>0.46 (0.20, 0.72)</td>
<td>0.46 (0.17)</td>
<td>0.46 (0.17)</td>
<td>0.46 (0.17)</td>
</tr>
<tr>
<td>$\sigma_B$ (%)</td>
<td>2.12 (2.00, 2.24)</td>
<td>2.12 (0.12)</td>
<td>2.12 (0.12)</td>
<td>2.12 (0.12)</td>
</tr>
<tr>
<td>$\sigma_S$ (%)</td>
<td>4.26 (3.99, 4.54)</td>
<td>4.26 (0.22)</td>
<td>4.26 (0.22)</td>
<td>4.26 (0.22)</td>
</tr>
<tr>
<td>$corr_{BS}$</td>
<td>0.10 (0.02, 0.18)</td>
<td>0.10 (0.06)</td>
<td>0.10 (0.06)</td>
<td>0.10 (0.06)</td>
</tr>
<tr>
<td>$skew_B$</td>
<td>0.20 (-0.06, 0.46)</td>
<td>0.20 (0.23)</td>
<td>0.04 (0.07)</td>
<td>0.02$^i$</td>
</tr>
<tr>
<td>$skew_S$</td>
<td>-0.64 (-1.06, -0.24)</td>
<td>-0.64 (0.35)</td>
<td>-0.63 (0.35)</td>
<td>-0.64 (0.35)</td>
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<tr>
<td>$coskew_{BS}$</td>
<td>-0.07 (-0.23, 0.10)</td>
<td>-0.30$^i$</td>
<td>-0.10 (0.04)</td>
<td>-0.07$^i$</td>
</tr>
<tr>
<td>$coskew_{SB}$</td>
<td>0.21 (0.01, 0.42)</td>
<td>0.44$^i$</td>
<td>0.26 (0.13)</td>
<td>0.21 (0.14)</td>
</tr>
<tr>
<td>$xkurt_B$</td>
<td>1.47 (0.90, 2.02)</td>
<td>0.28$^i$</td>
<td>0.03$^i$</td>
<td>0.01$^i$</td>
</tr>
<tr>
<td>$xkurt_S$</td>
<td>2.40 (0.54, 4.40)</td>
<td>1.33$^i$</td>
<td>1.28$^i$</td>
<td>1.33$^i$</td>
</tr>
<tr>
<td>$\psi$</td>
<td></td>
<td>0.66 (0.34)</td>
<td>0.41 (0.15)</td>
<td>0.36 (0.17)</td>
</tr>
<tr>
<td>$\delta_B$</td>
<td></td>
<td>0.47 (0.18)</td>
<td>0.28 (0.08)</td>
<td>0.22 (0.13)</td>
</tr>
<tr>
<td>$\delta_S$</td>
<td></td>
<td>-0.69 (0.13)</td>
<td>-0.68 (0.12)</td>
<td>-0.69 (0.13)</td>
</tr>
</tbody>
</table>
Table 2: Risk attitudes and optimal portfolios

The table presents detailed information about the optimal portfolio choice of specific investors. For the GDA investors in Panel A, $\gamma = 2$ and $\ell = 2$ are used, while $\kappa$ varies across columns. The investment horizon is one month. The distribution of asset returns is calibrated using the values reported in panel “GMM III” of Table 1. The log certainty equivalent, expected shortfall, and upside potential are in monthly percentage values. Panel B presents values for a comparable EU investor. The effective risk aversion, $\tilde{\gamma}$, of the investor is exactly the same as that of the GDA investor in the same column, but her implicit asymmetry aversion is the one implied by EU preferences. The disappointment probabilities ($\pi_W$) in Panel B are calculated using the corresponding threshold ($\ln \kappa + \eta$) reported in Panel A.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>0.96</th>
<th>0.97</th>
<th>0.98</th>
<th>0.99</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A – GDA investor</strong> ($\gamma = 2$ and $\ell = 2$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Effective risk aversion, $\tilde{\gamma}$</td>
<td>6.3</td>
<td>8.1</td>
<td>11.9</td>
<td>23.2</td>
<td>20.1</td>
<td>10.5</td>
<td>7.3</td>
<td>5.7</td>
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<tr>
<td>Implicit asymmetry aversion, $\tilde{\chi}$ ($\times 100$)</td>
<td>4.27</td>
<td>4.37</td>
<td>4.46</td>
<td>4.54</td>
<td>-4.27</td>
<td>-3.77</td>
<td>-3.32</td>
<td>-2.92</td>
</tr>
<tr>
<td>Disappointment threshold, $\ln \kappa + \eta$ (%)</td>
<td>-3.51</td>
<td>-2.51</td>
<td>-1.54</td>
<td>-0.57</td>
<td>1.46</td>
<td>2.51</td>
<td>3.54</td>
<td>4.56</td>
</tr>
<tr>
<td>Disappointment probability, $\pi_W$ (%)</td>
<td>1.9</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>89.2</td>
<td>90.6</td>
<td>91.9</td>
<td>92.9</td>
</tr>
<tr>
<td>Bond disapp. prob., $\pi_B$ (%)</td>
<td>2.9</td>
<td>7.7</td>
<td>16.8</td>
<td>30.7</td>
<td>67.4</td>
<td>82.8</td>
<td>92.3</td>
<td>97.1</td>
</tr>
<tr>
<td>Stock disapp. prob., $\pi_S$ (%)</td>
<td>14.1</td>
<td>19.2</td>
<td>25.6</td>
<td>33.3</td>
<td>52.8</td>
<td>63.6</td>
<td>73.5</td>
<td>81.8</td>
</tr>
<tr>
<td>Cash weight, $w_f$ (%)</td>
<td>13.2</td>
<td>33.2</td>
<td>54.3</td>
<td>76.5</td>
<td>71.6</td>
<td>45.8</td>
<td>22.3</td>
<td>0.6</td>
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<td>Bond weight, $w_B$ (%)</td>
<td>52.4</td>
<td>40.5</td>
<td>27.8</td>
<td>14.3</td>
<td>10.5</td>
<td>20.8</td>
<td>30.8</td>
<td>40.4</td>
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<tr>
<td>Stock weight, $w_S$ (%)</td>
<td>34.4</td>
<td>26.3</td>
<td>17.9</td>
<td>9.1</td>
<td>17.9</td>
<td>33.4</td>
<td>47.0</td>
<td>59.0</td>
</tr>
</tbody>
</table>

| MV fund weight, $\alpha^{MV}$ (%) | 89.0 | 68.5 | 46.9 | 24.1 | 27.7 | 53.0 | 76.3 | 97.8 |
| AV fund weight, $\alpha^{AV}$ (%) | 2.15 | 1.70 | 1.19 | 0.62 | -0.67 | -1.13 | -1.44 | -1.62 |

| **Panel B – Comparable EU investor** |
| Effective risk aversion, $\tilde{\gamma}$ | 6.3 | 8.1 | 11.9 | 23.2 | 20.1 | 10.5 | 7.3 | 5.7 |
| Implicit asymmetry aversion, $\tilde{\chi}$ ($\times 10^3$) | 0.37 | 0.40 | 0.43 | 0.47 | 0.46 | 0.42 | 0.38 | 0.35 |
| Disappointment probability, $\pi_W$ (%) | 3.5 | 3.9 | 4.3 | 4.7 | 93.1 | 93.8 | 94.5 | 95.1 |
| Cash weight, $w_f$ (%) | 11.2 | 31.6 | 53.2 | 76.0 | 72.4 | 47.1 | 23.9 | 2.4 |
| Bond weight, $w_B$ (%) | 43.9 | 33.8 | 23.2 | 11.9 | 13.7 | 26.2 | 37.7 | 48.2 |
| Stock weight, $w_S$ (%) | 44.9 | 34.5 | 23.6 | 12.1 | 13.9 | 26.7 | 38.5 | 49.4 |
| MV fund weight, $\alpha^{MV}$ (%) | 89.0 | 68.5 | 46.9 | 24.1 | 27.7 | 53.0 | 76.3 | 97.8 |
| AV fund weight, $\alpha^{AV}$ (%) | 0.18 | 0.15 | 0.11 | 0.06 | 0.07 | 0.13 | 0.17 | 0.20 |
Table 3: Cost of ignoring skewness

The table presents measures for the cost of ignoring return asymmetries. The preference parameters used are the same as in the corresponding columns of Table 2. The investment horizon is one month for all investors and the distribution of asset returns is calibrated using the values reported in panel “GMM III” of Table 1. The cost in absolute terms is measured as \((R - R')\), while in relative terms it is measured as \((R' - R_f) / (R - R_f)\), \(R\) being the annualized certainty equivalent of the optimal portfolio. An investor who ignores return asymmetry and chooses her optimal portfolio as if log asset returns were normally distributed with the same mean and variance-covariance matrix as the true distribution, chooses the suboptimal allocation \(w'\). \(R'\) is the annualized certainty equivalent of the suboptimal allocation under the true distribution.

<table>
<thead>
<tr>
<th>(\kappa)</th>
<th>0.96</th>
<th>0.97</th>
<th>0.98</th>
<th>0.99</th>
<th>1.01</th>
<th>1.02</th>
<th>1.03</th>
<th>1.04</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A – GDA investor ((\gamma = 2 \text{ and } \ell = 2))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relative cost, (\frac{R' - R_f}{R - R_f}) (%)</td>
<td>89.17</td>
<td>89.04</td>
<td>89.01</td>
<td>89.09</td>
<td>96.63</td>
<td>97.19</td>
<td>97.68</td>
<td>98.11</td>
</tr>
<tr>
<td>Absolute cost, (R - R' (\times 10^3))</td>
<td>2.677</td>
<td>2.120</td>
<td>1.478</td>
<td>0.765</td>
<td>0.350</td>
<td>0.540</td>
<td>0.619</td>
<td>0.624</td>
</tr>
<tr>
<td>Panel B – Comparable EU investor</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relative cost, (\frac{R' - R_f}{R - R_f}) (%)</td>
<td>99.96</td>
<td>99.95</td>
<td>99.94</td>
<td>99.93</td>
<td>99.93</td>
<td>99.95</td>
<td>99.96</td>
<td>99.96</td>
</tr>
<tr>
<td>Absolute cost, (R - R' (\times 10^3))</td>
<td>0.008</td>
<td>0.007</td>
<td>0.006</td>
<td>0.004</td>
<td>0.004</td>
<td>0.006</td>
<td>0.008</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Table 4: Asset allocations recommended by financial advisors

Panel A of the table presents assumptions regarding the distribution of monthly asset returns based on Table 2 (p. 185) of Canner et al. (1997). Note that Canner et al. (1997) do not report asset skewness, so we use the same values as used in the calibration exercise in the present paper. Panel B of the table presents the recommendations of four financial advisors. The first four columns are taken from Table 1 (p. 183) of Canner et al. (1997). The last two columns present the relative weights in the mean-variance fund \( \bar{w}^{\text{MV}} \) (\( \alpha^{\text{MV}} \)) and the asymmetry-variance fund \( \bar{w}^{\text{AV}} \) (\( \alpha^{\text{AV}} / \alpha^{\text{MV}} \)) for each portfolio.

### Panel A – Assumed asset returns

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std. dev.</th>
<th>Skew</th>
<th>Correlation with</th>
<th>( \bar{w}^{\text{MV}} )</th>
<th>( \bar{w}^{\text{AV}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Bonds</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cash</td>
<td>0.05%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonds</td>
<td>0.18%</td>
<td>2.9%</td>
<td>0.02</td>
<td>1.00</td>
<td>0.27</td>
<td>22.5</td>
</tr>
<tr>
<td>Stocks</td>
<td>0.75%</td>
<td>6.0%</td>
<td>-0.64</td>
<td>0.23</td>
<td>0.73</td>
<td>-21.5</td>
</tr>
</tbody>
</table>

### Panel B – Portfolio recommendations

<table>
<thead>
<tr>
<th>Advisor</th>
<th>Percent of portfolio (%)</th>
<th>( w_C )</th>
<th>( w_B )</th>
<th>( w_S )</th>
<th>( w_S \times 100 )</th>
<th>( \alpha^{\text{MV}} ) (%)</th>
<th>( \alpha^{\text{AV}} / \alpha^{\text{MV}} \times 100 )</th>
</tr>
</thead>
</table>
| Advisor A
| Conservative            | 50        | 30        | 20        | 150      | 49                 | 1.5             |
| Moderate              | 20        | 40        | 40        | 100      | 79                 | 1.1             |
| Aggressive           | 5         | 30        | 65        | 46       | 95                 | 0.2             |
| Advisor B
| Conservative            | 20        | 35        | 45        | 78       | 79                 | 0.8             |
| Moderate              | 5         | 40        | 55        | 73       | 94                 | 0.7             |
| Aggressive           | 5         | 20        | 75        | 27       | 95                 | -0.3            |
| Advisor C
| Conservative            | 50        | 30        | 20        | 150      | 49                 | 1.5             |
| Moderate              | 10        | 40        | 50        | 80       | 89                 | 0.8             |
| Aggressive           | 0         | 0         | 100       | 0        | 101                | -1.2            |
| Advisor D
| Conservative            | 20        | 40        | 40        | 100      | 79                 | 1.1             |
| Moderate              | 10        | 30        | 60        | 50       | 90                 | 0.3             |
| Aggressive           | 0         | 20        | 80        | 25       | 100                | -0.3            |
Figure 1: Conditional bond–stock correlations and expected stock shortfalls

Figure A plots conditional bond–stock correlations defined as $\text{Corr}(r_S, r_B | r_S < Q_S(q))$ if $q \leq 0.5$ and $\text{Corr}(r_S, r_B | r_S > Q_S(q))$ if $q > 0.5$, where $Q_S(q)$ denotes the $q$th quantile of the stock return distribution. Figure B plots the expected stock return shortfall defined as $E[r_S | r_S < Q_S(q)]$ and expressed in monthly percentages (%). Both figures display values estimated from the sample (Sample), values simulated from a normal distribution fitted to the data (Normal), and values simulated using model (9) fitted to the data using different GMM estimators (i.e., GMM I, GMM II, and GMM III).

A. Conditional correlations

B. Expected stock shortfalls
Figure 2: Optimal portfolios for different investors

The figure presents the relative weights in the mean-variance fund ($\alpha^{MV}$) and the asymmetry-variance fund ($\alpha^{AV}/\alpha^{MV}$) corresponding to optimal portfolios for investors with different preferences. All curves start at the same point corresponding to the investor for whom $\gamma = 2$ and $\ell = 0$. The line corresponding to the EU investor shows the effect of increasing $\gamma$ from 2 to 30. Increasing $\gamma$ leads to higher effective risk aversion and therefore means moving left along the horizontal axis. The other curves correspond to disappointment-averse investors with different $\kappa$ values (see the legend) and show the effect of increasing $\ell$ from 0 to 3 while keeping $\gamma$ fixed at 2. Increasing $\ell$ leads to higher effective risk aversion and therefore means moving left along the horizontal axis. The investment horizon is one month and the distribution parameters for the asset returns are given in column “GMM III” of Table 1.
Figure 3: Optimal portfolios for different levels of stock skewness
The figure presents optimal portfolio weights in the stock and the bond for an EU investor ($\gamma = 6.3$) and two GDA investors (one with $\gamma = 2$, $\ell = 2$, and $\kappa = 0.96$, and the other with $\gamma = 2$, $\ell = 2$, and $\kappa = 1.04$). The investment horizon is one month and the distribution parameters, except $\delta_S$, are given in column “GMM III” of Table 1. The parameter $\delta_S$ is varied along the horizontal axis leading to different stock skewness values. The vertical line corresponds to our benchmark calibration with a stock skewness of –0.64.
Figure 4: The effect of increasing the investment horizon

Figure A shows how the optimal portfolio changes with the investment horizon if returns are IID. The distribution parameters for the one-period returns are given in column “GMM III” of Table 1, while the $H$-period parameters are calculated according to (24). The figure shows the relative weight in the asymmetry-variance fund ($\alpha_{AV}/\alpha_{MV}$) for the EU ($\ell = 0$) investor and two GDA investors (one for whom $\kappa < 1$ and the other for whom $\kappa > 1$). The preference parameters are chosen so that the effective risk aversion is $\tilde{\gamma} = 5$ for all investors and horizons. Figures B to D compare the IID assumption to the case when the return generating model (9) is fit to returns aggregated over $H = 1, \ldots, 12$ months. Figure B shows the stock’s skewness when returns are aggregated over $H$ months (round markers) and in the IID case (solid line). Figures C and D show the optimal stock weight and bond/stock allocation ratio, respectively, for the EU investor and the GDA investor with $\kappa < 1$ in the IID case, and for the same GDA investor with the estimated $H$-period returns.
Online Appendix to
“Asymmetries and Portfolio Choice”

Magnus Dahlquist    Adam Farago    Roméo Tédongap*

June 30, 2015

Abstract

We examine the portfolio choice of an investor with generalized disappointment aversion preferences who faces returns described by a normal-exponential model. We derive a three-fund separation strategy: the investor allocates wealth to a risk-free asset, a standard mean-variance efficient fund, and an additional fund reflecting return asymmetries. The optimal portfolio is characterized by the investor’s endogenous effective risk aversion and implicit asymmetry aversion. We find that disappointment aversion is associated with much larger asymmetry aversion than are standard preferences. Our model explains patterns in popular portfolio advice and provides a reason for shifting from bonds to stocks as the investment horizon increases.

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1 Accounting for idiosyncratic skewness

In the paper, we use the following model to describe the log returns on $N$ risky assets:

$$r_t = \mu - \sigma \circ \delta + (\sigma \circ \delta) \varepsilon_{0,t} + \left(\sigma \circ \sqrt{\iota - \delta \circ \delta}\right) \circ \varepsilon_t ,$$  \hspace{1cm} (1)

where $\mu$, $\sigma$, and $\delta$ are $N$-dimensional vectors, $\iota$ is a vector of ones, and $\circ$ denotes the Schur product (element-wise product) of vectors. The scalar $\varepsilon_{0,t}$ is a common shock across all assets that follows an exponential distribution with a rate parameter equal to one. The $N$-dimensional vector $\varepsilon_t$ represents asset specific shocks and has a multivariate normal distribution, independent of $\varepsilon_{0,t}$, with standard normal marginal densities and correlation matrix $\Psi$. The model in (1) assumes that the return asymmetry is generated by an exposure to a common skewed factor. The model can be extended to account for idiosyncratic asset skewness by adding asset-specific exponential shocks:

$$r_t = \mu - \sigma \circ \left(\delta + \lambda\right) + (\sigma \circ \delta) \varepsilon_{0,t} + (\sigma \circ \lambda) \circ \xi_t + \left(\sigma \circ \sqrt{\iota - \delta \circ \delta - \lambda \circ \lambda}\right) \circ \varepsilon_t ,$$  \hspace{1cm} (2)

where $\lambda$ is an $N$-dimensional vector and $\xi_t$ is a vector of independent asset specific shocks, each of them having an exponential distribution with a rate parameter equal to one. It is straightforward to see that if $\lambda = 0$, we are back to the return generating model in (1). The moments of the asset returns are given by

$$E(r_{i,t}) = \mu_i , \hspace{0.5cm} Var(r_{i,t}) = \sigma_i^2 , \hspace{0.5cm} Skew(r_{i,t}) = 2 \left(\delta_i^3 + \lambda_i^3\right) , \hspace{0.5cm} Coskew(r_{i,t}, r_{j,t}) = 2\delta_i^2\delta_j .$$  \hspace{1cm} (3)

Note that the parameter $\lambda_i$ appears in the formula for the skewness of asset $i$, but not in the formula for the co-skewness with other assets. Hence, $\lambda_i$ is only important for idiosyncratic skewness.

In the following, we illustrate how our main results change if the extended return gen-
erating model (2) is used instead of (1). The parameters from (2) can be estimated by matching the moments in (3). If there are two risky assets, the return generating model has 9 parameters. We use the same data as for the main calibration of the paper (the two risky assets are a bond and a stock). We calibrate the parameters by matching the two means, two volatilities, correlation, two skewnesses, and two coskewnesses. The calibrated means ($\mu$) and volatilities ($\sigma$) are the same as in Table 1 of the paper, corresponding to the GMM III estimation. The remaining parameters are

$$\psi = 0.42, \quad \delta_B = 0.22, \quad \delta_S = -0.68, \quad \lambda_B = 0.45, \quad \text{and} \quad \lambda_S = -0.17. \quad (4)$$

First, note that the asset exposures the common skewed shock barely change when we take into account idiosyncratic skewness (the $\delta$ values are almost the same as for the GMM III case in Table 1 of the paper). Second, the systemic asymmetry component dominates the idiosyncratic one in the case of the stock ($\delta_S$ is greater in magnitude than $\lambda_S$), while it is the other way round for the bond.

We use numerical results to illustrate that the main conclusions are similar using the return generating model (2) to those obtained using model (1). We generate $T = 500,000$ realizations both from models (1) and (2). Then the optimal portfolio weights are chosen using numerical optimization to maximize the certainty equivalent $R(W)$ that is endogenously defined by

$$\theta U (\mathcal{R}(W)) = E [U(W)] - \ell E [(U(\kappa\mathcal{R}(W)) - U(W)) I(W < \kappa\mathcal{R}(W))]. \quad (5)$$
The graph below shows optimal portfolios for different preference parameters. The horizontal axis corresponds to the stock weight ($w_{Stock}$), while the vertical axis represents the relative bond weight ($w_{Bond}/w_{Stock}$). The curves, similar to Figure 2 in the paper, correspond to disappointment-averse investors with different $\kappa$ values and show the effect of increasing $\ell$ from 0 to 3, while keeping $\gamma$ fixed at 2. Moreover, the markers correspond to specific values of $\ell$ to facilitate comparison across the return generating models ($\bigcirc$ is $\ell = 0.2$, $\Box$ is $\ell = 0.5$, $\triangle$ is $\ell = 1$, and $\Diamond$ is $\ell = 2$). As it can be seen in the graph, optimal portfolios do not change much if we switch from the return generating model (1) to model (2). The optimal stock weight is almost identical, while the relative bond weight is somewhat higher when idiosyncratic skewness is taken into account. However, the difference is small.
2 Approximation of the portfolio log return

The second-order Taylor approximation of the portfolio log return proposed by Campbell and Viceira (2002) is

\[ r_W \approx r_f + w^\top \left( r_t - r_{ft} + \frac{1}{2} \sigma^2 \right) - \frac{1}{2} w^\top \Sigma w , \]

(6)

where \( w \) denotes the vector of portfolio weights, \( \Sigma \) is the variance-covariance matrix of \( r_t \), and \( \sigma^2 \) denotes the vector of the diagonal elements of \( \Sigma \). Extending the approximation to a third-order one yields

\[ r_W \approx r_f + w^\top \left( r_t - r_{ft} + \frac{1}{2} \sigma^2 + \frac{1}{6} s \right) - \frac{1}{2} w^\top (\Sigma + S) w + \frac{1}{3} (w \otimes w)^\top \Upsilon w , \]

(7)

with

\[ s = E \left[ (r_t - \mu) \circ (r_t - \mu) \circ (r_t - \mu) \right] \]
\[ S = E \left[ ((r_t - \mu) \circ (r_t - \mu)) (r_t - \mu)^\top \right] \]
\[ \Upsilon = E \left[ ((r_t - \mu) \otimes (r_t - \mu)) (r_t - \mu)^\top \right] \]

(8)

where \( \circ \) denotes the element-wise product of two vectors and \( \otimes \) denotes their Kronecker-product. Note that the difference between the approximations in (6) and (7) affects only the mean of \( r_W \), since the higher order moments depend only on the \( w^\top r_t \) part of the approximations. In other words, when the risky asset returns are described by the model in (1), then for both approximations, the portfolio log return can be expressed as

\[ r_W = \mu_W - \sigma_W \delta_W + (\sigma_W \delta_W) \varepsilon_{0,t} + \left( \sigma_W \sqrt{1 - \delta_W^2} \right) \varepsilon_{W,t} , \]

(9)
\[
\mu_W^{(2)} = r_f + w^\top \left( \mu - r_f \mu + \frac{1}{2} \sigma^2 \right) - \frac{1}{2} w^\top \Sigma w , \\
\mu_W^{(3)} = r_f + w^\top \left( \mu - r_f \mu + \frac{1}{2} \sigma^2 + \frac{1}{6} s \right) - \frac{1}{2} w^\top (\Sigma + S) w + \frac{1}{3} (w \otimes w)^\top \Upsilon w , \\
\sigma^2_W = w^\top \Sigma w , \\
\delta_W = \frac{w^\top (\sigma \circ \delta)}{\sigma_W} ,
\]

where \( \mu_W^{(2)} \) and \( \mu_W^{(3)} \) are the means for the second- and third-order approximations, respectively. Note that \( \sigma^2_W \) and \( \delta_W \) are the same for both approximations.

To assess how well the approximations work, we simulate \( T = 10,000,000 \) realizations of the return generating process (1) to get the simulated series \( r_{ts} \). The parameters for the return generating model are taken from the benchmark calibration in the paper (GMM III in Table 1). Then, for a given vector of risky portfolio weights \( w \), the portfolio log return is

\[
r_{W,t}^s = \log \left( \sum_{i=1}^{N} w_i \exp (r_{t,i}^s) + \left( 1 - \sum_{i=1}^{N} w_i \right) \exp (r_f) \right) .
\]

The following graphs show how the first three moments of the simulated and approximated portfolio log return change when the stock weight, \( w_{stock} \) is varied between 0 and 1, and the bond weight is set to \( w_{bond} = 1 - w_{stock} \).

A. Mean

B. Std. Dev.
Panel A shows that means of the simulated and approximated portfolio log returns are almost identical, regardless of the order of the approximation. Panel B shows that the standard deviations of the simulated and approximated portfolio log returns are also very close to each other. Finally, Panel C shows that the skewness of the approximated returns is lower (larger in magnitude) than the skewness of the simulated returns for intermediate $w_{stock}$ values. However, the difference is small.
3 Additional derivations for Proposition 2.1

The return generating model for the $H$-period log returns between dates $t$ and $t+H$ is

$$
\begin{align*}
    r_{t,t+H} &= \mu - \sqrt{H} (\sigma_H \circ \delta) + (\sigma_H \circ \delta) \varepsilon_{0,t,t+H} + \left(\sigma_H \circ \sqrt{1 - \delta \circ \delta}\right) \circ \varepsilon_{t,t+H}, \quad (12)
\end{align*}
$$

where,

$$
\begin{align*}
    \varepsilon_{0,t,t+H} &\sim \Gamma \left(H, 1/\sqrt{H}\right), \\
    \varepsilon_{t,t+H} &\sim N(0, \Psi). \quad (13)
\end{align*}
$$

To simplify the notation, in the Online Appendix we use $r_H$ to denote $H$-period returns instead of $r_{t,t+H}$, and we drop the $H$ subscript from the parameters $\mu_H$ and $\sigma_H$. The $H$-period portfolio log return is denoted as $r_{W,H}$. We begin by the general case $H \geq 1$. Later we consider the special case $H = 1$, corresponding to Proposition 2.1 in the paper.

**Analytical expression for $M(u, v; H, x)$**

Define

$$
M(u, v; H, x) \equiv E \left[ \exp \left( u r_{W,H} + v^\top r_H \right) I (r_{W,H} < x) \right]. \quad (14)
$$

The return generating model can be written as

$$
\begin{align*}
    r_H &= \mu - \sqrt{H} \sigma \circ \delta + (\sigma \circ \delta) \varepsilon_{0,H} + \left(\sigma \circ \sqrt{1 - \delta \circ \delta}\right) \circ \varepsilon_H \\
    &= \mu - \sqrt{H} a + a \varepsilon_{0,H} + \Lambda \varepsilon_H, \quad (15)
\end{align*}
$$

with $a \equiv \sigma \circ \delta$ and $\Lambda \equiv Diag \left(\sigma_1 \sqrt{1 - \delta^2_1}, \sigma_2 \sqrt{1 - \delta^2_2}, \ldots, \sigma_N \sqrt{1 - \delta^2_N}\right)$. For any $b \in \mathbb{R}$ and $c \in \mathbb{R}^N$

$$
\begin{align*}
    b + c^\top r_H &= b + c^\top \mu - \sqrt{H} c^\top a + c^\top a \varepsilon_{0,H} + \sqrt{c^\top \Lambda \Psi \Lambda c} - \frac{c^\top \Lambda \varepsilon_H}{\sqrt{c^\top \Lambda \Psi \Lambda c}}, \\
    \text{where } \frac{c^\top \Lambda \varepsilon_H}{\sqrt{c^\top \Lambda \Psi \Lambda c}} &\sim N(0, 1).
\end{align*}
$$
First, equation (16) with \( b = \mathbf{w}^\top (-r_f t + \frac{1}{2} \sigma^2) - \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w} + r_f \) and \( c = \mathbf{w} \) implies that the portfolio log return can be written as

\[
    r_{W,H} = \mu_W - \sqrt{H} a_W + a_W \varepsilon_{0,H} + \sigma_W \sqrt{1 - \delta^2_w} \varepsilon_{W,H},
\]

(17)

with

\[
    \mu_W = \mathbf{w}^\top \left( \mu - r_f t + \frac{1}{2} \sigma^2 \right) - \frac{1}{2} \sigma^2_W + r_f \\
    \sigma^2_W = \mathbf{w}^\top \left( (\Lambda \Psi \Lambda) + aa^\top \right) \mathbf{w} \\
    a_W = \mathbf{w}^\top a \\
    \delta_W = a_W / \sigma_W \\
    \varepsilon_{W,H} = \frac{w^\top \Lambda \varepsilon_H}{\sqrt{w^\top (\Lambda \Psi \Lambda) w}} \sim N(0,1).
\]

Second, equation (16) also implies that for \( u \in \mathbb{R} \) and \( v \in \mathbb{R}^N \),

\[
    ur_{W,H} + v^\top r_H = \bar{\mu}(u,v) - \sqrt{H} \bar{a}(u,v) + \bar{a}(u,v) \varepsilon_{0,H} + \bar{\sigma}(u,v) \sqrt{1 - \bar{\delta}^2(u,v)} \varepsilon_H(u,v)
\]

(19)

where

\[
    \bar{\mu}(u,v) = u \mu_W + v^\top \mu \\
    \bar{\sigma}^2(u,v) = (uw + v)^\top \left( (\Lambda \Psi \Lambda) + aa^\top \right) (uw + v) \\
    \bar{a}(u,v) = (uw + v)^\top a \\
    \bar{\delta}(u,v) = \bar{a}(u,v) / \bar{\sigma}(u,v) \\
    \varepsilon_H(u,v) = \frac{(uw + v)^\top \Lambda \varepsilon_H}{\sqrt{(uw + v)^\top (\Lambda \Psi \Lambda) (uw + v)}} \sim N(0,1).
\]

(20)

Define

\[
    \rho(u,v) = (uw + v)^\top (\Lambda \Psi \Lambda) w.
\]

(21)
Lemma 1 Let

\[ r_1 = \mu_1 - \sqrt{H} \sigma_1 \delta_1 + \sigma_1 \delta_1 \epsilon_0 + \sigma_1 \sqrt{1 - \delta_1^2} \epsilon_1 \quad \text{and} \]
\[ r_2 = \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \sigma_2 \delta_2 \epsilon_0 + \sigma_2 \sqrt{1 - \delta_2^2} \epsilon_2 \]

where the variable \( \epsilon_0 \sim \Gamma \left( H, \frac{1}{\sqrt{H}} \right) \), and \( \epsilon_1 \) and \( \epsilon_2 \) are two standard normal variables independent of \( \epsilon_0 \) with correlation \( \psi \). Then

\[ E \left[ \exp (r_2) I (r_1 < x) \right] = \exp \left( \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \frac{\sigma_2^2 (1 - \delta_2^2)}{2} \right) H^{\frac{\mu_2}{2}} B \left( H - 1 \right), \quad (22) \]

where

\[ B (0) = \frac{1}{b_2} \left[ \Phi (b_0) + C (0) \right] \]
\[ B (\tau) = \frac{1}{b_2} \left[ B (\tau - 1) + \frac{1}{\tau!} C (\tau) \right] \]
\[ C (0) = \begin{cases} 
\exp \left( \frac{2b_0 b_1 b_2 + b_2^3}{2b_1^3} \right) \Phi \left( \frac{-b_0 b_1 + b_2}{b_1} \right) & \text{if } b_1 > 0 \\
- \exp \left( \frac{2b_0 b_1 b_2 + b_2^3}{2b_1^3} \right) \Phi \left( \frac{b_0 b_1 + b_2}{b_1} \right) & \text{if } b_1 < 0 
\end{cases} \]
\[ C (1) = \frac{1}{b_1} \phi (b_0) - \frac{b_2 + b_0 b_1}{b_1^2} C (0) \]
\[ C (\tau) = \frac{1}{b_1^2} \left[ -(b_0 b_1 + b_2) C (\tau - 1) + (\tau - 1) C (\tau - 2) \right] \]

and

\[ b_0 \equiv \frac{x - \mu_1 + \sqrt{H} \sigma_1 \delta_1}{\sigma_1 \sqrt{1 - \delta_1^2}} - \psi \sigma_2 \sqrt{1 - \delta_2^2} , \]
\[ b_1 \equiv \frac{-\delta_1}{\sqrt{1 - \delta_1^2}} , \]
\[ b_2 \equiv \sqrt{H} - \sigma_2 \delta_2 . \]

and \( \Phi (\cdot) \) denotes the standard normal cumulative distribution function.
Proof of Lemma 1. We want to find an analytical expression for

\[ M(x) \equiv E[\exp(r_2) I(r_1 < x)]. \]  

(23)

Conditioning on the exponential shock and using the law of iterated expectations

\[
M(x) = E \left[ E \left[ \exp \left( \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \sigma_2 \delta_2 \varepsilon_0 + \sigma_2 \sqrt{1 - \delta_2^2} \right) I \left( \mu_1 - \sqrt{H} \sigma_1 \delta_1 + \sigma_1 \delta_1 \varepsilon_0 + \sigma_1 \sqrt{1 - \delta_1^2} \varepsilon_1 < x \right) \mid \varepsilon_0 \right] \right]
\]

(24)

where

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} \sim N \left( \begin{pmatrix}
\mu_1 - \sqrt{H} \sigma_1 \delta_1 \\
\mu_2 - \sqrt{H} \sigma_2 \delta_2
\end{pmatrix}, \begin{pmatrix}
\sigma_1^2 \left( 1 - \delta_1^2 \right) & \psi \sigma_1 \sqrt{1 - \delta_1^2} \sigma_2 \sqrt{1 - \delta_2^2} \\
\psi \sigma_1 \sqrt{1 - \delta_1^2} \sigma_2 \sqrt{1 - \delta_2^2} & \sigma_2^2 \left( 1 - \delta_2^2 \right)
\end{pmatrix} \right).
\]

(25)

Since the above vector is bivariate normal,

\[ E[\exp(Y) I(X < \bar{x})] = \exp \left( \mu_Y + \frac{\sigma_Y^2}{2} \right) \Phi \left( \frac{\bar{x} - \mu_X - \sigma_{XY}}{\sigma_X} \right), \]

(26)

and consequently

\[ E[\exp(Y) I(X < x - \sigma_1 \delta_1 \varepsilon_0) \mid \varepsilon_0] = \exp \left( \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \frac{\sigma_2^2 \left( 1 - \delta_2^2 \right)}{2} \right) \Phi \left( b_0 + b_1 \varepsilon_0 \right), \]

(27)

with

\[
b_0 \equiv \frac{x - \mu_1 + \sqrt{H} \sigma_1 \delta_1}{\sigma_1 \sqrt{1 - \delta_1^2}} - \psi \sigma_2 \sqrt{1 - \delta_2^2},
\]
\[
b_1 \equiv \frac{-\delta_1}{\sqrt{1 - \delta_1^2}},
\]
\[
b_2 \equiv \sqrt{H} - \sigma_2 \delta_2.
\]
Substituting the above result in (24) we arrive at

\[ M(x) = \exp \left( \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \frac{\sigma_2^2 (1 - \delta_2^2)}{2} \right) E[\exp (\sigma_2 \delta_2 \varepsilon_0) \Phi (b_0 + b_1 \varepsilon_0)] . \] (28)

We need to find a formula for \( E[\exp (\sigma_2 \delta_2 \varepsilon_0) \Phi (b_0 + b_1 \varepsilon_0)] \). Let us define

\[
B(\tau) \equiv \frac{1}{\tau!} \int_0^\infty \exp (-b_2 z) z^\tau \Phi (b_0 + b_1 z) \, dz ,
\]

\[
C(\tau) \equiv \int_0^\infty \exp (-b_2 z) z^\tau b_1 \phi (b_0 + b_1 z) \, dz .
\]

Then, using the fact that \( \varepsilon_{0,H} \sim \Gamma \left( H, 1/\sqrt{H} \right) \),

\[
E[\exp (\sigma_2 \delta_2 \varepsilon_0) \Phi (b_0 + b_1 \varepsilon_0)] = \int_0^\infty \exp (\sigma_2 \delta_2 z) \frac{H^\mu}{(H-1)!} \exp \left(-\sqrt{Hz} \right) z^{H-1} \Phi (b_0 + b_1 z) \, dz
\]

\[
= H^\mu \frac{1}{(H-1)!} \int_0^\infty \exp (-b_2 z) z^{H-1} \Phi (b_0 + b_1 z) \, dz
\]

\[
= H^\mu B(H-1) .
\] (29)

Using integration by parts, it can be shown that

\[
B(0) = \frac{1}{b_2} [\Phi (b_0) + C(0)] ,
\]

\[
B(\tau) = \frac{1}{b_2} \left[ B(\tau - 1) + \frac{1}{\tau!} C(\tau) \right] .
\] (30)

The above recursion allows us to calculate the value of \( B(\tau) \) for arbitrary value of \( \tau \geq 0 \) if
we know the value of \( C(\tau) \). Again, using integration by parts, we can show that

\[
C(0) = \begin{cases} 
\exp \left( \frac{2 b_0 b_1 b_2 + b_2^2}{2 b_1^2} \right) \Phi \left( - \frac{b_0 b_1 + b_2}{b_1} \right) & \text{if } b_1 > 0 \\
- \exp \left( \frac{2 b_0 b_1 b_2 + b_2^2}{2 b_1^2} \right) \Phi \left( \frac{b_0 b_1 + b_2}{b_1} \right) & \text{if } b_1 < 0
\end{cases}
\]

\[
C(1) = \frac{1}{b_1} \phi(b_0) - \frac{b_2 + b_0 b_1}{b_1^2} C(0)
\]

\[
C(\tau) = \frac{1}{b_1^2} \left[ -(b_0 b_1 + b_2) C(\tau - 1) + (\tau - 1) C(\tau - 2) \right].
\]

Also note that if \( x = \infty \)

\[
E \left[ \exp(r_2 I(r_1 < x)) \right] = \exp \left( \mu_2 - \sqrt{H} \sigma_2 \delta_2 + \frac{\sigma_2^2 (1 - \delta_2^2)}{2} \right) \frac{H^{\mu}}{b_1^2 H^2} \tag{32}
\]

\[
\square
\]

Lemma 1 can be applied to arrive at the formula for \( M(u;v;H,x) \) by setting \( r_1 = r_W \) and \( r_2 = u r_{W,H} + v^T r_H \). Consequently,

\[
M(u;v;H,x) = \exp \left( \mu(u,v) - \sqrt{H} \ddot{a}(u,v) + \frac{\ddot{a}^2(u,v) - \dot{a}^2(u,v)}{2} \right) H^{\mu} B \left( u;v;H-1,x,\sqrt{H} \right)
\]

with

\[
B \left( u;v;\tau,x,\sqrt{H} \right) = \frac{1}{\sqrt{H} - \ddot{a}(u,v)} \left[ B \left( u;v;\tau-1,x,\sqrt{H} \right) + \frac{1}{\tau!} C \left( u;v;\tau,x,\sqrt{H} \right) \right]
\]

\[
B \left( u;v;0,x,\sqrt{H} \right) = \frac{1}{\sqrt{H} - \ddot{a}(u,v)} \left[ \Phi \left( b_0(u,v) \right) + C \left( u;v;0,x,\sqrt{H} \right) \right]
\]

\[
C \left( u;v;\tau,x,\sqrt{H} \right) = \frac{1}{b_1^2} \left[ \left( \ddot{a}(u,v) - \sqrt{H} - b_0(u,v) \right) C \left( u;v;\tau-1,x,\sqrt{H} \right) + (\tau - 1) C \left( u;v;\tau-2,x,\sqrt{H} \right) \right]
\]

\[
C \left( u;v;0,x,\sqrt{H} \right) = \begin{cases} 
\exp \left( (\sqrt{H} - \ddot{a}(u,v)) \frac{\sqrt{H} - \ddot{a}(u,v) + b_0(u,v) b_1}{2 b_1^2} \right) \Phi \left( - \frac{\sqrt{H} - \ddot{a}(u,v) + b_0(u,v) b_1}{b_1} \right) & \text{if } b_1 > 0 \\
- \exp \left( (\sqrt{H} - \ddot{a}(u,v)) \frac{\sqrt{H} - \ddot{a}(u,v) + b_0(u,v) b_1}{2 b_1^2} \right) \Phi \left( \frac{\sqrt{H} - \ddot{a}(u,v) + b_0(u,v) b_1}{b_1} \right) & \text{if } b_1 < 0
\end{cases}
\]

\[
C \left( u;v;1,x,\sqrt{H} \right) = \frac{1}{b_1} \phi(b_0(u,v)) - \frac{\sqrt{H} - \ddot{a}(u,v) + b_0(u,v) b_1}{b_1^2} C \left( u;v;0,x,\sqrt{H} \right)
\]

\[
(34)
\]
and

\[ b_0 (u, v) \equiv \frac{x - \mu_W + \sqrt{H} a_W - \bar{\rho} (u, v)}{\sigma_W \sqrt{1 - \delta_W^2}} \]

\[ b_1 \equiv \frac{-\delta_W}{\sqrt{1 - \delta_W^2}} \]  

(35)

Note also that if \( x = \infty \), then

\[ M (u, v; H, \infty) = \exp \left( \bar{\mu} (u, v) - \sqrt{H} \bar{a} (u, v) + \frac{\bar{\sigma}^2 (u, v) - \bar{a}^2 (u, v)}{2} \right) \frac{H^\frac{H}{2}}{\left( \sqrt{H} - \bar{a} (u, v) \right)^H} \]  

(36)

The optimal portfolio rule for \( H \geq 1 \)

Using the formula in (33) it can be shown that

\[ \frac{M' (u, v; H, x)}{M (u, v; H, x)} = \mu - \sqrt{H} a + \Sigma (uw + v) - \bar{a} (u, v) a + \frac{B_2' (u, v; H - 1, x, \sqrt{H})}{B (u, v; H - 1, x, \sqrt{H})}, \]  

(37)

where \( B_2' (u, v; H - 1, x, \sqrt{H}) \) is the partial derivative of \( B (u, v; H - 1, x, \sqrt{H}) \) with respect to the second argument \( v \). It can be shown that for any \( \tau \geq 0 \), the value of \( B_2' (u, v; H - 1, x, \sqrt{H}) \) can be calculated recursively as

\[ B_2' (u, v; \tau, x, \sqrt{H}) = \frac{1}{\sqrt{H - \bar{a} (u, v)}} \left( \xi_{a, \tau}^B + \xi_{\Sigma, \tau}^B \Sigma w + B (u, v; \tau, x, \sqrt{H}) a \right) \]  

(38)

with

\[ \xi_{a, \tau}^B = \frac{\xi_{a, \tau-1}^B}{\sqrt{H - \bar{a} (u, v)}} + \xi_{\Sigma, \tau}^B \frac{B (u, v; \tau - 1, x, \sqrt{H})}{\sqrt{H - \bar{a} (u, v)}} \]

\[ \xi_{\Sigma, \tau}^B = \frac{\xi_{\Sigma, \tau-1}^B}{\sqrt{H - \bar{a} (u, v)}} + \xi_{\Sigma, \tau}^B \frac{B (u, v; \tau - 1, x, \sqrt{H})}{\sqrt{H - \bar{a} (u, v)}} \]  

(39)

\[ \xi_{a, 0}^B = \xi_{a, 0}^C + \frac{\phi (b_0 (u, v)) a_W}{\sigma_W \sqrt{1 - \delta_W^2}} \]

\[ \xi_{\Sigma, 0}^B = \xi_{\Sigma, 0}^C - \frac{\phi (b_0 (u, v))}{\sigma_W \sqrt{1 - \delta_W^2}} \]  

(40)
Substituting (38) into (37) and setting $u = 1 - \gamma$, $b = 0$, and $x = \ln \kappa + \eta$, we arrive at

$$\frac{M'_2 (1 - \gamma, 0; H, \ln \kappa + \eta)}{M (1 - \gamma, 0; H, \ln \kappa + \eta)} = \mu + (1 - \gamma + \tilde{\gamma}) \Sigma w + \left( \frac{1 - H + (1 - \gamma)^2 a_w^2}{\sqrt{H} - (1 - \gamma) a_w} + \tilde{\chi} \right) a$$

with

$$\tilde{\gamma} = \frac{1}{\sqrt{H} - (1 - \gamma) a_w} B \left( 1 - \gamma, 0; H - 1, \ln \kappa + \eta, \sqrt{H} \right)$$

$$\tilde{\chi} = \frac{1}{\sqrt{H} - (1 - \gamma) a_w} B \left( 1 - \gamma, 0; H - 1, \ln \kappa + \eta, \sqrt{H} \right)$$

Substituting (38) into (37) and setting $u = 1 - \gamma$, $b = 0$, and $x = \ln \kappa + \eta$, we arrive at
The $\xi$ values in the above formula can be calculated recursively using the formulas in (39) to (43), while the $B$ values can be calculated with the formulas in (34). Note also that

$$
\frac{M'_2 (1 - \gamma, 0; H, \infty)}{M (1 - \gamma, 0; H, \infty)} = \mu + (1 - \gamma) \Sigma w + \frac{(1 - \gamma)^2 a^2_{W}}{\sqrt{H - (1 - \gamma) a_W}} a
$$

(46)

Substituting (44) and (46) into

$$
G'_1 (w, \eta) = \frac{M'_2 (1 - \gamma, 0; H, \infty)}{M (1 - \gamma, 0; H, \infty)} + \frac{\nu}{1 - \nu} \frac{M'_2 (1 - \gamma, 0; H, \ln \kappa + \eta)}{M (1 - \gamma, 0; H, \ln \kappa + \eta)} + \frac{1}{1 - \nu} \left( -r_{f} + \frac{1}{2} \sigma^2 - \Sigma w \right).
$$

(47)

setting (47) to zero, and solving for $w$, we can arrive at the optimal portfolio rule

$$
w = \frac{1}{\tilde{\gamma}} \left( \Sigma^{-1} \left( \mu - r_{f} + \frac{1}{2} \sigma^2 \right) + \tilde{\chi} \Sigma^{-1} (\sigma \circ \delta) \right),
$$

(48)

with

$$
\tilde{\gamma} \equiv \gamma - \nu \tilde{\gamma}
$$

$$
\tilde{\chi} \equiv \frac{(1 - \gamma)^2 a^2_{W}}{\sqrt{H - (1 - \gamma) a_W}} + \nu \frac{1 - H}{\sqrt{H - (1 - \gamma) a_W}} + \nu \tilde{\chi}
$$

(49)

Note that $\ell = 0$ implies $\nu = 0$.

The special case of $H = 1$

Using $u = 1 - \gamma$, $v = \infty$, $H = 1$, and $x = \ln \kappa + \eta$ in (33) leads to

$$
M (1 - \gamma, 0; 1, \ln \kappa + \eta) = \exp \left( (1 - \gamma) (\mu_W - \sigma_W \delta_W) + \frac{(1 - \gamma)^2 \sigma^2_W}{2} \frac{1 - \delta^2_W}{2c^2_1} \right) \frac{\Phi (c_0) + C}{c_2}
$$

(50)

with

$$
C \equiv C (1 - \gamma, 0; 0, \ln \kappa + \eta, 1) = \begin{cases} 
\exp \left( \frac{c^2_2 + 2c_0 c_1 c_2}{2c^2_1} \right) \Phi \left( -\frac{c_2 + c_0 c_1}{c_1} \right) & \text{if } c_1 > 0 \\
- \exp \left( \frac{c^2_2 + 2c_0 c_1 c_2}{2c^2_1} \right) \Phi \left( \frac{c_2 + c_0 c_1}{c_1} \right) & \text{if } c_1 < 0
\end{cases}
$$

(51)
and
\[
c_0 \equiv b_0 (u, 0) = \frac{\ln \kappa + \eta - \mu_W + \sqrt{H} \sigma_W \delta_W - (1 - \gamma) \sigma_W^2 (1 - \delta_W^2)}{\sigma_W \sqrt{1 - \delta_W^2}}
\]
\[
c_1 \equiv b_1 = \frac{-\delta_W}{\sqrt{1 - \delta_W^2}}
\]
\[
c_2 \equiv 1 - (1 - \gamma) \sigma_W \delta_W.
\] (52)

Note also that
\[
M (1 - \gamma, 0; 1, \infty) = \exp \left( (1 - \gamma) (\mu_W - \sigma_W \delta_W) + \frac{(1 - \gamma)^2 \sigma_W^2 (1 - \delta_W^2)}{2} \right) \frac{1}{c_2}.
\] (53)

Then substituting \( H = 1 \) into (44) and using the formulas in (34), (40), (43), and (45), we arrive at
\[
\frac{M' (1 - \gamma, 0; 1, \ln \kappa + \eta)}{M (1 - \gamma, 0; 1, \ln \kappa + \eta)} = \mu + \left( 1 - \gamma + \frac{\xi_{\Sigma,0}^B}{\Phi (c_0) + C} \right) \Sigma_w + \left( \frac{(1 - \gamma)^2 \sigma_W^2 \delta_W^2}{c_2} + \frac{\xi_{a,0}^B}{\Phi (c_0) + C} \right) (\sigma \circ \delta),
\] (54)

with
\[
\begin{align*}
\xi_{a,0}^B &= \exp \left( \frac{c_2^2 + 2 c_0 c_1 c_2}{2 c_1^2} \right) \frac{\phi}{\sqrt{c_1^2}} \left( -c_2 + c_0 c_1 \right) \left( -c_2 - c_2 + c_0 c_1 \right) \\
&\quad + \exp \left( \frac{c_2^2 + 2 c_0 c_1 c_2}{2 c_1^2} \right) \frac{\phi}{\sqrt{c_1^2}} \left( -c_2 + c_0 c_1 \right) \left( \frac{1}{c_1} + 1 \right) - c_1 \phi (c_0) \\
\xi_{\Sigma,0}^B &= \exp \left( \frac{c_2^2 + 2 c_0 c_1 c_2}{2 c_1^2} \right) \frac{\phi}{\sqrt{c_1^2}} \left( -c_2 + c_0 c_1 \right) \left( \frac{c_2}{\sigma_W \delta_W} \right) \\
&\quad + \exp \left( \frac{c_2^2 + 2 c_0 c_1 c_2}{2 c_1^2} \right) \frac{\phi}{\sqrt{c_1^2}} \left( -c_2 + c_0 c_1 \right) \left( \frac{1}{\sigma_W \sqrt{1 - \delta_W^2}} \right) - \frac{\phi (c_0)}{\sigma_W \sqrt{1 - \delta_W^2}}
\end{align*}
\] (55)

Also, substituting \( H = 1 \) into (46) leads to
\[
\frac{M' (1 - \gamma, 0; 1, \infty)}{M (1 - \gamma, 0; 1, \infty)} = \mu + (1 - \gamma) \Sigma_w + \frac{(1 - \gamma)^2 \sigma_W^2 \delta_W^2}{c_2} (\sigma \circ \delta)
\] (56)

Substituting (54) and (56) in (47), setting it to zero, and solving for \( w \), we can arrive at the optimal portfolio rule
\[
w = \frac{1}{\gamma} \left( \Sigma^{-1} \left( \mu - r_f t + \frac{1}{2} \sigma^2 \right) + \tilde{X} \Sigma^{-1} (\sigma \circ \delta) \right),
\] (57)
with

$$\tilde{\gamma} = \gamma - \nu \frac{\xi_{B,0}}{\Phi(c_0) + C}$$

$$\tilde{\chi} = \frac{(1 - \gamma)^2 \sigma_W^2 \delta_W^2}{1 - (1 - \gamma) \sigma_W \delta_W} + \nu \frac{\xi_{B,0}}{\Phi(c_0) + C}. \tag{58}$$
4 Asset pricing implications

Suppose there are $K$ investors denoted $k = 1, 2, \ldots, K$, and investor $k$ has wealth $W_k$. Denote by $W = \sum_{k=1}^K W_k$ the total market capitalization, and let $w$ be the market portfolio. We have shown that the optimal solution to the portfolio choice problem for an investor $k$ may be written

$$w_k = \frac{1}{\tilde{\gamma}_k} \Sigma^{-1} \left( \mu - \nu r_f + \frac{1}{2} \sigma^2 \right) + \frac{\tilde{\chi}_k}{\tilde{\gamma}_k} \Sigma^{-1} (\sigma \circ \delta),$$

where $\tilde{\gamma}_k$ and $\tilde{\chi}_k$ are the effective risk aversion and the implicit asymmetry aversion respectively. Aggregating asset demands across investors yields

$$\sum_{k=1}^K w_k W_k = wW,$$

implying that

$$w = \frac{1}{\tilde{\gamma}} \Sigma^{-1} \left( \mu - \nu r_f + \frac{1}{2} \sigma^2 \right) + \frac{\tilde{\chi}}{\tilde{\gamma}} \Sigma^{-1} (\sigma \circ \delta),$$

where

$$\frac{1}{\tilde{\gamma}} = \sum_{k=1}^K \pi_k \frac{1}{\tilde{\gamma}_k} \quad \text{and} \quad \frac{\tilde{\chi}}{\tilde{\gamma}} = \sum_{k=1}^K \pi_k \frac{\tilde{\chi}_k}{\tilde{\gamma}_k}, \quad \text{and} \quad \pi_k = \frac{W_k}{W}. \quad (62)$$

Note that equation (61) is equivalent to

$$\mu - \nu r_f + \frac{1}{2} \sigma^2 = \tilde{\gamma} \Sigma w - \tilde{\chi} (\sigma \circ \delta).$$

Pre-multiply equation (63) by the market portfolio $w^\top$, keeping in mind the Campbell and Viceira (2002) approximation of a portfolio log return, and obtain

$$w^\top \left( \mu - \nu r_f + \frac{1}{2} \sigma^2 \right) = \tilde{\gamma} w^\top \Sigma w - \tilde{\chi} w^\top (\sigma \circ \delta),$$

or equivalently

$$\mu W - \nu r_f + \frac{1}{2} \sigma_W^2 = \tilde{\gamma} \sigma_W^2 - \tilde{\chi} \sigma_w \delta_W. \quad (65)$$
where \( \mu_W, \sigma_W^2 \) and \( \delta_W \) are the mean, the variance, and the asymmetry of the market portfolio log return.

Next, consider a portfolio \( w^* \) with a zero covariance and equal asymmetry with the market portfolio. It follows that

\[
\begin{align*}
    w^{*\top} \Sigma w &= 0 \\
    w^{*\top} (\sigma \circ \delta) &= \sigma_W \delta_W.
\end{align*}
\]

There are 2 equations with \( N \geq 2 \) unknowns. Thus the orthogonal portfolio \( w^* \) exists but may not be unique.

Pre-multiply equation (63) by the orthogonal portfolio weights \( w^{*\top} \) and obtain

\[
 w^{*\top} \left( \mu - \nu r_f + \frac{1}{2} \sigma^2 \right) = \tilde{\gamma} w^{*\top} \Sigma w - \tilde{\chi} w^{*\top} (\sigma \circ \delta),
\]

or equivalently, given equation (66),

\[
 \mu^*_W - r_f + \frac{1}{2} \sigma^*_W = -\tilde{\chi} \sigma_W \delta_W,
\]

where \( \mu^*_W \) and \( \sigma^*_W \) and the mean and the variance of the orthogonal portfolio log return.

For an asset \( i \), we have from equation (63)

\[
 \mu_i - r_f + \frac{1}{2} \sigma_i^2 = \tilde{\gamma} \sigma_i w - \tilde{\chi} \sigma_i \delta_i
\]

where \( \mu_i, \sigma_i^2 \) and \( \delta_i \) are the mean, the variance, and the asymmetry of asset \( i \), and where \( \sigma_{iw} \) is the covariance of asset \( i \) with the market portfolio.

Combining the equations (65), (68) and (69), we can eliminate \( \tilde{\gamma} \) and \( \tilde{\chi} \), and obtain

\[
 \mu_i - r_f + \frac{1}{2} \sigma_i^2 = \frac{\sigma_{iw}}{\sigma_W^2} \left( \mu_W - r_f + \frac{1}{2} \sigma_W^2 \right) + \left( \frac{\sigma_i \delta_i}{\sigma_W \delta_W} - \frac{\sigma_{iw}}{\sigma_W^2} \right) \left( \mu_W - r_f + \frac{1}{2} \sigma_W^2 \right).
\]

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\begin{equation}
\begin{aligned}
w^* &= \left[ \frac{\sigma \circ \delta \Sigma^{-1} (\sigma \circ \delta)}{\sigma^2_W \delta^2_W} - \frac{1}{\sigma^2_W} \right]^{-1} \left[ \frac{\Sigma^{-1} (\sigma \circ \delta)}{\sigma_W \delta_W} - \frac{w}{\sigma^2_W} \right]
\end{aligned}
\end{equation}

(71)

Note that the portfolio $w^*$ satisfies the conditions in equation (66). Let $\sigma_{iW}^*$ denote the covariance of asset $i$ with the orthogonal portfolio. With $w^*$ in equation (71), one can easily verify that

\begin{equation}
\begin{aligned}
\sigma^2_W &= \left[ \frac{\sigma \circ \delta \Sigma^{-1} (\sigma \circ \delta)}{\sigma^2_W \delta^2_W} - \frac{1}{\sigma^2_W} \right]^{-1} \text{ and } \frac{\sigma_{iW}^*}{\sigma^2_W} = \frac{\sigma_i \delta_i}{\sigma_W \delta_W} - \frac{\sigma_{iW}}{\sigma^2_W}.
\end{aligned}
\end{equation}

(72)

Equation (70) is then equivalent to

\begin{equation}
\begin{aligned}
\mu_i - r_f + \frac{1}{2} \sigma_i^2 &= \beta_{iW} \left( \mu_W - r_f + \frac{1}{2} \sigma^2_W \right) + \beta_{iW}^* \left( \mu_{iW} - r_f + \frac{1}{2} \sigma^2_{iW} \right),
\end{aligned}
\end{equation}

(73)

where

\begin{equation}
\begin{aligned}
\beta_{iW} &\equiv \frac{\sigma_{iW}}{\sigma^2_W} \text{ and } \beta_{iW}^* \equiv \frac{\sigma_{iW}^*}{\sigma^2_{iW}}
\end{aligned}
\end{equation}

(74)

define the exposures (or betas) of asset $i$ to the market portfolio and the orthogonal portfolio.

Barberis and Huang (2008) also consider a setting where investors have asymmetric preferences, namely cumulative prospect theory (CPT) preferences, and there is a positively skewed asset return. They find that the asset’s own skewness is priced. Our asset pricing implications show the same: Expected returns are related to the asset’s own skewness ($\delta_i$) through its exposure to the orthogonal portfolio ($\beta_{iW}^*$).
5 Relation with prospect theory

We can rewrite the certainty equivalent of an investor with generalized disappointment aversion (GDA) preferences

\[ \theta U(R(W)) = E[U(W)] - \ell E[(U(\kappa R(W)) - U(W)) I(W < \kappa R(W))], \tag{75} \]

as

\[ U(R(W)) = E[Z(W) U(W)], \tag{76} \]

where

\[ Z(W) = \frac{1 + \ell I(W < \kappa R(W))}{\theta + \ell \kappa^{1-\gamma} E[I(W < \kappa R(W))]}, \tag{77} \]

\[ U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \]

and where \( \gamma \geq 0, \ell \geq 0 \) and \( \kappa > 0 \). Equation (76) shows that the GDA investor maximizes a weighted sum of utility of all possible terminal wealth outcomes, each outcome being assigned a decision weight that is proportional to its likelihood, and the coefficient of proportionality is given by equation (77). If \( \ell = 0 \), the coefficient of proportionality is constant and equal to one for all outcomes. The investor is then maximizing expected utility (EU) and the weight assigned to each outcome corresponds to its likelihood. In contrast, if \( \ell > 0 \) the coefficient of proportionality is \( 1 + \ell \) times larger for disappointing outcomes relative to non-disappointing outcomes. EU with power utility is a special case of GDA and the utility \( U \) is concave everywhere.

Cumulative prospect theory (CPT) preferences are not defined on terminal wealth, but on terminal wealth relative to a reference wealth (i.e., on gains and losses). Thus EU is not a special case. CPT investors also maximize a weighted sum of utility of all possible gain-loss outcomes, each outcome being assigned a decision weight that is a nonlinear function of its
cumulative probability. Also, the utility is concave on gains and convex on losses, and the weighting scheme differs for gains and losses. Formally, the CPT investor maximizes

$$E[H(W^e)V(W^e)],$$

where

$$H(W^e) = h'_+(1 - F(W^e))I(W^e \geq 0) + h'_-(F(W^e))I(W^e < 0)$$

$$h_+(X) = \frac{X^\gamma}{(X^\gamma + (1 - X)\gamma)^{1/\gamma}}$$ \text{ and } $$h_-(X) = \frac{X^\delta}{(X^\delta + (1 - X)\delta)^{1/\delta}}$$

$$V(W^e) = V_+(W^e)I(W^e \geq 0) + V_-(W^e)I(W^e < 0)$$

$$V_+(X) = X^\alpha$$ \text{ and } $$V_-(X) = -\lambda(-X)^\beta,$$

where $0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1, 0 < \delta < 1, \lambda > 1$, $F$ is the cumulative distribution function of excess terminal wealth $W^e \equiv W - W_f$, and where $h'$ is the first-order derivative of $h$.

CPT and GDA settings are similar in that both overweight low probability events. Regarding the CPT preference for positive skewness, it is potentially the same as GDA($\kappa < 1$) aversion to negative skewness. Barberis and Huang (2008) emphasize CPT preference for lotteries (upper tail gain); we emphasize GDA($\kappa < 1$) aversion to lower tail loss. However, there is some resemblance between the CPT and GDA settings. A GDA($\kappa > 1$) investor prefers high upside potential, something positive skewness may contribute to, but there are other contributors such as high expected return, volatility and kurtosis.
6 Relation to gain-loss ratios

Consider the terminal wealth

\[ W = W_0 R_W = W_0 \left( w^\top (R - \iota R_f) + R_f \right), \]

and work with gross returns instead of log returns as we do in the main paper. If we take the derivative of equation (75) with respect to the portfolio weight, within which we set the derivative of the certainty equivalent equal to zero, then we obtain the first-order condition

\[ E \left[ U' (W) \left( 1 + \ell I \left( W < \kappa R (W) \right) \right) (R - \iota R_f) \right] = 0, \tag{80} \]

where \( U' \) is the first-order derivative of \( U \). Equation (80) may also be written as

\[ \frac{E^* \left[ (R_i - R_f) I \left( W \geq \kappa R (W) \right) \right]}{E^* \left[ (R_f - R_i) I \left( W < \kappa R (W) \right) \right]} = 1 + \ell, \tag{81} \]

where \( E^* \) denotes the expectation operator under the risk-adjusted density

\[ \frac{U' (W)}{E \left[ U' (W) \right]}. \]

The left-hand side of equation (81) is a gain-loss ratio similar to Bernardo and Ledoit (2000). Practitioners often refer to a similar ratio as the omega ratio, in which the expectation operator is the actual one and not the risk-adjusted one. The difference is that, to the contrary of the gain-loss ratio and the omega ratio, the event conditional to which expectations are measured is not asset-specific but common to all assets. The first-order condition (81) says that all assets must have the same gain-loss ratio and that the gain-loss ratio is constant and equal to \( 1 + \ell \). With \( \ell > 0 \), GDA investors want a gain-loss ratio larger than that of the EU investor. GDA investors with \( \kappa < 1 \) achieve that by choosing allocations such as to
reduce their expected downside losses, while investors with $\kappa > 1$ achieve that by choosing allocations such as to increase their expected upside gains.
References

